MATH 348 - Advanced Engineering Mathematics February 4, 2008
Homework 4, Spring 2008 **Due**: February 13, 2008

Continuous Fourier Transform - Signal Processing - Green's Functions

If a function is not periodic nor can it be periodically extended then the function has no Fourier series representation. However, in this case the function can have a Fourier integral representation. Analysis of the Fourier integral representation[1] reveals the complex Fourier transform pairs:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega \tag{1}$$

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \tag{2}$$

We can connect (1)-(2) to the complex Fourier series. If a function is periodic then there exists a representation of the function in the countably-infinite basis of imaginary-exponential functions. In this case the coefficients[2] of this expansion are given by an integral whose limits of integration are bounded.[3] These coefficients quantify the amplitude of oscillation for each discrete frequency of oscillation.[4] In the case of (1) we have that the function f has a representation in the uncountably-infinite basis of imaginary-exponential functions. In this case the coefficients are given by an integral (2) whose limits of integration are unbounded.[5] These coefficients quantify the amplitude of oscillation for each continuous frequency of oscillation.

1. Now, we want to connect all of this to sections 11.7 and 11.8 in our text. We have that the Fourier integral represents functions in the sine/cosine basis without requiring the function to be periodic. Now, what role does symmetry play in this representation? With minimal work we see that if a function is even/odd then the Fourier integral reduces to a Fourier cosine/sine integral.[6] As with our original derivation of transform, if we look at the interplay between the coefficient functions $A(\omega)/B(\omega)$ and $f(x)$ we find the transform pairs:

$$f_c(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{f}(\omega)\cos(\omega x) d\omega \qquad \hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x)\cos(\omega x) dx \tag{3}$$

$$f_s(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{f}(\omega)\sin(\omega x) d\omega \qquad \hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x)\sin(\omega x) dx \tag{4}$$

We call (3) the Fourier cosine transform pair and, surprisingly, we call (4) the Fourier sine transform.

(a) Show that $f_c(x)$ and $\hat{f}_c(\omega)$ are even functions and that $f_s(x)$ and $\hat{f}_s(\omega)$ are odd functions.[7]

(b) Show that if we assume that $f(x)$ is an even function then (1)-(2) defines the transform pair given by (3). Also, show that if $f(x)$ is an odd function then (1)-(2) defines the transform pair given by (4). [8]

Given,

$$f(x) = \begin{cases} A, & 0 < x < a \\ 0, & \text{otherwise} \end{cases}, \qquad A, a \in \mathbb{R}^+. \tag{5}$$

(c) On the same graph plot the even and odd extensions of f.

(d) Find the Fourier cosine and sine transforms of f.

(e) Using the Fourier cosine transform show that $\displaystyle\int_{-\infty}^{\infty} \frac{\sin(\pi\omega)}{\pi\omega} d\omega = 1$.

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[1] See classnotes or Kreyszig pgs. 518-519

[2] also called weights

[3] Recall that our derivation lead to $c_n = \frac{1}{2\pi}\int_{-L}^{L} f(x)e^{i\omega_n x}dx$ where $\omega_n = \frac{n\pi}{L}$.

[4] That is, for each ω_n there is a corresponding c_n where $|c_n|^2$ is a measure of the power of the sinusoids associated with ω_n.

[5] In this case the behavior of f must be known everywhere instead of on the interval $(-L, L)$.

[6] Kreyszig pg. 511

[7] Thus, if an input function has an even or odd symmetry then the transformed function shares the same symmetry.

[8] Thus, if an input function has symmetry then the Fourier transform is real-valued.

2. Calculate the following transforms:

   (a) $\mathfrak{F}\{f\}$ where $f(x) = \delta(x - x_0)$, $x_0 \in \mathbb{R}$.[9]

   (b) $\mathfrak{F}\{f\}$ where $f(x) = e^{-k_0|x|}$, $k_0 \in \mathbb{R}^+$.

   (c) $\mathfrak{F}^{-1}\left\{\hat{f}\right\}$ where $\hat{f}(\omega) = \frac{1}{2}\left(\delta(\omega + \omega_0) + \delta(\omega - \omega_0)\right)$, $\omega_0 \in \mathbb{R}$.

   (d) $\mathfrak{F}^{-1}\left\{\hat{f}\right\}$ where $\hat{f}(\omega) = \frac{1}{2}\left(\delta(\omega + \omega_0) - \delta(\omega - \omega_0)\right)$, $\omega_0 \in \mathbb{R}$.

3. The convolution h of two functions f and g is defined as[10],

$$h(x) = (f * g)(x) = \int_{-\infty}^{\infty} f(p)g(x-p)dp = \int_{-\infty}^{\infty} f(x-p)g(p)dp. \tag{6}$$

   (a) Show that $\mathfrak{F}\{f * g\} = \sqrt{2\pi}\mathfrak{F}\{f\}\mathfrak{F}\{g\}$.

   (b) Find the convolution $h(x) = (f * g)(x)$ where $f(x) = \delta(x - x_0)$ and $g(x) = e^{-x}$.

4. Read the introductory paragraph of following websites `http://en.wikipedia.org/wiki/Autocorrelation` and `http://en.wikipedia.org/wiki/Cross-correlation` and respond to the following:

   (a) Write down an integral definition of cross-correlation and auto-correlation.

   (b) Compare and contrast cross-correlation and convolution in terms of independent random variables described by the probability distributions f and g.

   (c) Describe an application of cross-correlation to signal processing.

5. Given the ODE,

$$y' + y = f(x), \quad -\infty < x < \infty. \tag{7}$$

   Let $f(x) = \delta(x)$ and then:

   (a) Calculate the frequency response associated with (7).[11]

   (b) Calculate the Green's function associated with (7).

   (c) Using convolution find the steady-state solution to the (7).

[9] Here the δ is the so-called Dirac, or continuous, delta function. This isn't a function in the true sense of the term but instead what is called a generalized function. The details are unimportant and in this case we care only that this Dirac-delta *function* has the property $\int_{-\infty}^{\infty} \delta(x - x_0)f(x)dx = f(x_0)$. For more information on this matter consider `http://en.wikipedia.org/wiki/Dirac_delta_function`. To drive home that this *function* is strange, let me spoil the punch-line. The sampling function $f(x) = \mathrm{sinc}(ax)$ can be used as a definition for the Delta *function* as $a \to 0$. So can a bell-curve probability distribution. Yikes!

[10] Here wee keep the same notation as Kreysig pg. 523

[11] this is often called the steady-state transfer function