MATH332 - July 27, 2009
Exam II - 50 Points

NAME: Key

In order to receive full credit, SHOW ALL YOUR WORK. Full credit will be given only if all reasoning and work is provided. When applicable, please enclose your final answers in boxes.

1. (10 Points) True/False and Short Response

(a) Mark each statement True or False. No justification is needed.

i. The set of all n^{th}−degree polynomials, $p(t)$, such that $p(0) = 3$, forms a vector subspace of $\mathbb{P}_n$.

False, $p(t) = 0$ is not in the space

ii. The null space of $\mathbf{A}_{m \times n}$ is a subspace of $\mathbb{R}^m$.

True

iii. The column space of $\mathbf{A}_{m \times n}$ is the set of all solutions to $\mathbf{Ax} = \mathbf{b}$.

False, a sol$_n$ to $A\vec{x} = \vec{b}$ is $\vec{x} \in \mathbb{R}^m$ the column space is a subspace of $\mathbb{R}^n$.

iv. The columns of an $n \times n$ invertible matrix forms a basis for $\mathbb{R}^n$.

True, IMT

v. If $\mathbf{v}_1$ and $\mathbf{v}_2$ are two linearly independent eigenvectors then they must correspond to two different eigenvalues.

False, See Examples in Class for counter Example.

(b) Please respond to **both** of the following. Provide justification for your position.

i. Could a 6×9 matrix have a two-dimensional null space?

Note $\dim \text{Row} A \le 6 \Rightarrow 0 \le \dim \text{Col} A \le 6$

Since $\text{Rank } A + \dim \text{Nul} A = 9 \Rightarrow$ If $\dim \text{Nul} A = 2 \Rightarrow \text{Rank } A = 7$ which is impossible.

$\Rightarrow$ ~~dim Nul A < 3~~

ii. Suppose that $\mathbf{P}_{\mathcal{B}}$ is a change of coordinates matrix. List three properties of $\mathbf{P}_{\mathcal{B}}$.

P_B by definition has columns which are the Basis for ~~a~~ $\mathbb{R}^n$. Thus by IMT,

1) P_B^{-1} Exists
2) $P_B \vec{x} = \vec{0}$ has only the trivial sol$_n$
3) $\det(P_B) \neq 0$

and so on...

2. (10 Points) Quickies:

(a) Determine the dimension of the vector space formed by the span of each of the three sets.

$$S_1 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \end{bmatrix} \right\}, \quad S_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}, \quad S_3 = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right\}$$

2 2 2

(b) Given,

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}. \tag{1}$$

Find one eigenvalue of $\mathbf{A}$.

$A\vec{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \Rightarrow \lambda = 1$

(c) Given,

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{2}$$

what is the dimension of the null-space, column-space and row-space of $\mathbf{A}$?

dim Nul A = 2

dim Col A = dim Row A = 3

(d) Given,

$$\mathbf{A} = \begin{array}{c} \begin{matrix} \vec{a}_1 & \vec{a}_2 & \end{matrix} \\ \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} \end{array}, \tag{3}$$

find one eigenvalue of $\mathbf{A}$.

note $2\vec{a}_1 = \vec{a}_2 \Rightarrow$ Col. of A are lin. ind.

$\Rightarrow \det(A) = 0 \Rightarrow \lambda = 0$ is an Eigenvalue of A.

3. (10 Points) Proofs:

(a) Show that if two matrices $\mathbf{A}$ and $\mathbf{B}$, are similar, $\mathbf{A} = \mathbf{PBP}^{-1}$, then they have the same eigenvalues.

$$\det(A - \lambda I) = \det(PBP^{-1} - \lambda PP^{-1}) =$$
$$= \det(P[B - \lambda I]P^{-1}) = \det(P)\det(B - \lambda I)\det(P^{-1}) =$$
$$= \det(B - \lambda I)$$

$\Rightarrow A, B$ have same char poly. $\Rightarrow$ Same Evalues

(b) If the set of all $n \times n$ matrices forms a vector space then show that the subset defined by all $n \times n$ symmetric matrices, $\mathbf{A} = \mathbf{A}^T$, forms a vector subspace.

1) $A = 0$ is s.t. $A^T = A$

3,2) Let A_1, A_2 be in this set and $\alpha, \beta \in \mathbb{R}$

Then
$(\alpha A_1 + \beta A_2)^T = \alpha A_1^T + \beta A_2^T = \alpha A_1 + \beta A_2$ which is in the space.

Thus, we have a vector space

4. (10 Points) Given,

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}. \tag{4}$$

Find a basis for the null-space, column-space and row-space of $\mathbf{A}$.

$$A \sim \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 0 & 0 & 1 & -1 & 8 \\ 0 & 0 & 1 & -1 & 4 \\ 0 & 0 & 2 & -2 & 13 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 0 & 0 & 1 & -1 & 8 \\ 0 & 0 & 0 & 0 & \cancel{12}\,-4 \\ 0 & 0 & 0 & 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B_{ColA} = \left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 2 \\ 8 \end{bmatrix} \right\}$$

$\Rightarrow x_5 = 0$

$x_4 = x_3$

$x_1 = -4x_2 - 2x_3 = -4x_2 - 2x_4$

$$B_{RowA} = \left\{ [1\ 4\ 0\ 2\ 0], [0\ 0\ 1\ -1\ 0], [0\ 0\ 0\ 0\ 1] \right\}$$

$$\vec{x} = \begin{bmatrix} -4x_2 - 2x_4 \\ x_2 \\ x_4 \\ x_4 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow B_{NulA} = \left\{ \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

5. (10 Points) Given,

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}. \quad (A-\lambda I) = \begin{bmatrix} -\lambda & 0 & 2 \\ 0 & 2-\lambda & 0 \\ 2 & 0 & -\lambda \end{bmatrix} \tag{5}$$

Using diagonalization calculate $\mathbf{A}^4$.

$$\det(A-\lambda I) = -\lambda(2-\lambda)(-\lambda) + 2\left(-2(2-\lambda)\right) =$$
$$= (2-\lambda)(\lambda^2-4) = 0 \quad \lambda_1 = -2,\ \lambda_2 = 2,\ \lambda_3 = 2$$

$$[A-\lambda_1 I|0] = \left[\begin{array}{ccc|c} 2 & 0 & 2 & 0 \\ 0 & 4 & 0 & 0 \\ 2 & 0 & 2 & 0 \end{array}\right] \Rightarrow \begin{matrix} x_1 = -x_3 \\ x_2 = 0 \end{matrix} \quad \vec{x}^{(1)} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad [A-\lambda_2 I|0] = \left[\begin{array}{ccc|c} -2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & -2 & 0 \end{array}\right]$$

$$\Rightarrow P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\Rightarrow \begin{matrix} x_1 = x_3 \\ x_2 \in \mathbb{R} \end{matrix} \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} x_2$$

$$[P|I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & \phi & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1/2 & 0 & 1/2 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 0 & -1/2 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1/2 & 0 & 1/2 \end{array}\right]$$

$$\Rightarrow A^4 = PD^4P^{-1} = P\begin{bmatrix} (-2)^4 & 0 & 0 \\ 0 & 2^4 & 0 \\ 0 & 0 & 2^4 \end{bmatrix}P^{-1} = 2^4 PIP^{-1} = 2^4 I = \begin{bmatrix} 16 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 16 \end{bmatrix}$$

6. (Extra Credit) Please choose **one** of the following:

(a) Find a basis for the set of all vectors of the form $\begin{bmatrix} a-2b+5c \\ 2a+5b-8c \\ -a-4b+7c \\ 3a+b+c \end{bmatrix}$, where $a, b, c \in \mathbb{R}$.

(b) Suppose that $\mathbf{A}_{n\times n}$ has $n-$distinct eigenvalues such that $|\lambda_i| < 1$ for $i = 1, 2, 3, \ldots, n$. Show that $\lim_{k\to\infty} \mathbf{A}^k$ is the zero matrix.

a) Note

$$\begin{bmatrix} a-2b+5c \\ 2a+5b-8c \\ -a+-4b+7c \\ 3a+b+c \end{bmatrix} = a\underbrace{\begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}}_{\vec{v}_1} + \underbrace{\begin{bmatrix} -2 \\ 5 \\ -4 \\ 1 \end{bmatrix}}_{\vec{v}_2} b + c\underbrace{\begin{bmatrix} 5 \\ -8 \\ 7 \\ 1 \end{bmatrix}}_{\vec{v}_3}$$

$\Rightarrow \vec{v}_3 = \vec{v}_1 + -2\vec{v}_2,\ \vec{v}_2 \neq \alpha\vec{v}_1$

$\Rightarrow \vec{v}_1, \vec{v}_2$ are lin. indep.

So these can be used as a basis for Vectors of this form.

b) If A has n-distinct Evalues then A is diagonalizable thus,

$\lim_{k\to\infty} A^k = \lim_{k\to\infty} PD^kP^{-1}$ where $\lim_{k\to\infty}[D^k]_i = \lim d_i^k =$
$= \lim_{k\to\infty} \lambda_i^k = 0$

By assumption.

Thus $\lim_{k\to\infty} D^k = 0$

$\Rightarrow A^k \to 0$ as $k \to \infty$.