

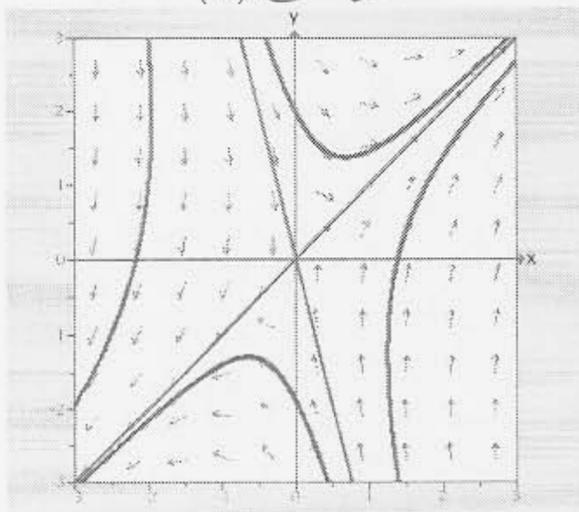
For full credit, you must show all work and box answers.

1. Match each of the following matrices with its possible phase portrait. Hint: Find the eigenvalues.

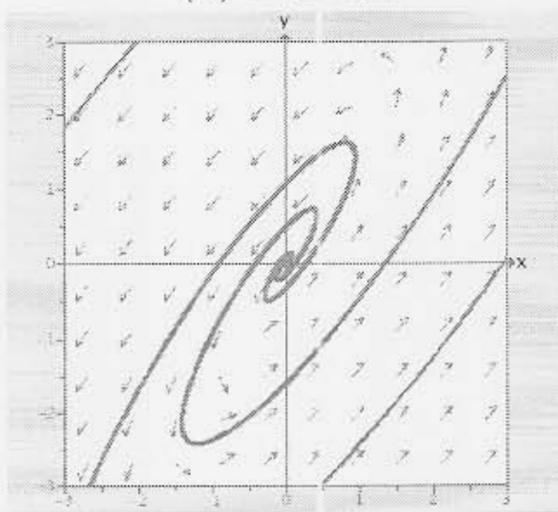
(i) $A = \begin{pmatrix} +1 & 1 \\ -1 & +3 \end{pmatrix}$ (ii) $A = \begin{pmatrix} 3 & -2 \\ 4 & -2 \end{pmatrix}$

(iii) $A = \begin{pmatrix} -1 & 2 \\ -2 & 4 \end{pmatrix}$ (iv) $A = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}$

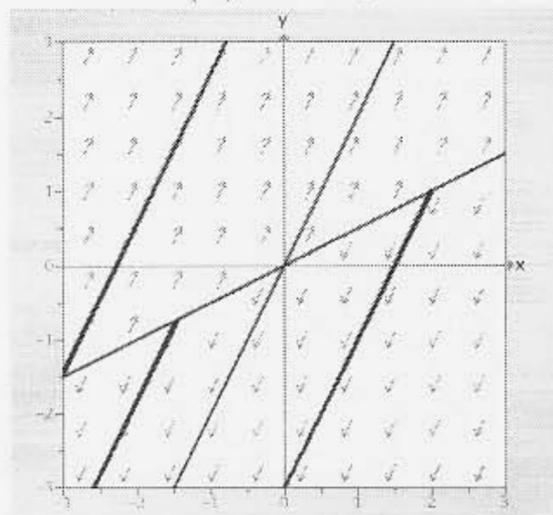
(A) (iv)



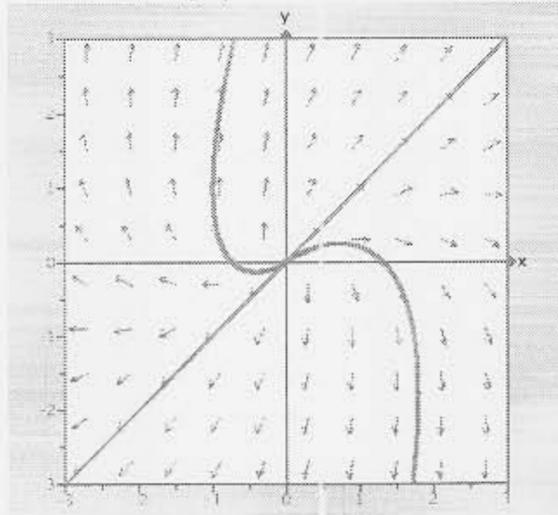
(B) (ii)



(C) (iii)



(D) (i)



(i) $A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$

$\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 1 \\ -1 & 3-\lambda \end{pmatrix} = 0$

$(1-\lambda)(3-\lambda) + 1 = 0$

$\lambda^2 - 4\lambda + 4 = 0$

$(\lambda - 2)^2 = 0, \lambda_1 = \lambda_2 = 2, \text{ Repeated}$

(iii) $A = \begin{pmatrix} -1 & 2 \\ -2 & 4 \end{pmatrix}$

$\det(A - \lambda I) = \det \begin{pmatrix} -1-\lambda & 2 \\ -2 & 4-\lambda \end{pmatrix} = 0$

$(-1-\lambda)(4-\lambda) + 4 = 0$

$\lambda^2 - 3\lambda = 0$

$\lambda(\lambda - 3) = 0$

$(\lambda_1 = 0, \lambda_2 = 3, \text{ zero eigenvalue})$

(ii) $A = \begin{pmatrix} 3 & -2 \\ 4 & -2 \end{pmatrix}$

$\det(A - \lambda I) = \det \begin{pmatrix} 3-\lambda & -2 \\ 4 & -2-\lambda \end{pmatrix} = 0$

$(3-\lambda)(-2-\lambda) + 8 = 0$

$\lambda^2 - \lambda + 2 = 0$

$\lambda = \frac{1 \pm \sqrt{1-8}}{2} = \frac{1 \pm \sqrt{7}i}{2}$

Spiral source

(iv) $A = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}$

$\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{pmatrix} = 0$

$(1-\lambda)(-2-\lambda) - 4 = 0$

$\lambda^2 + \lambda - 2 = 0$

$(\lambda + 2)(\lambda - 1) = 0$

Saddle, $\lambda_1 = -2, \lambda_2 = 1$

2. Given the system

$$\begin{aligned} \frac{dx}{dt} &= x - 2y \\ \frac{dy}{dt} &= 3x - 4y \end{aligned}$$

$$A = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}, \quad A - \lambda I = \begin{pmatrix} 1-\lambda & -2 \\ 3 & -4-\lambda \end{pmatrix}$$

(a) Find the general solution.

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ (1-\lambda)(-4-\lambda) + 6 &= 0 \\ \lambda^2 + 3\lambda + 2 &= 0 \\ (\lambda + 2)(\lambda + 1) &= 0 \\ \lambda_1 &= -2, \lambda_2 = -1 \end{aligned}$$

$$\lambda_1 = -2: (A - \lambda_1 I) \vec{v}_1 = \vec{0}$$

$$\begin{pmatrix} 3 & -2 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$3x_1 - 2y_1 = 0$$

$$y_1 = \frac{3x_1}{2}, \quad x_1 = \alpha$$

$$\begin{pmatrix} \alpha \\ \frac{3\alpha}{2} \end{pmatrix}, \quad \text{Let } \alpha = 2: \vec{v}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\lambda_2 = -1: (A - \lambda_2 I) \vec{v}_2 = \vec{0}$$

$$\begin{pmatrix} 2 & -2 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$2x_2 - 2y_2 = 0$$

$$y_2 = x_2, \quad x_2 = \alpha$$

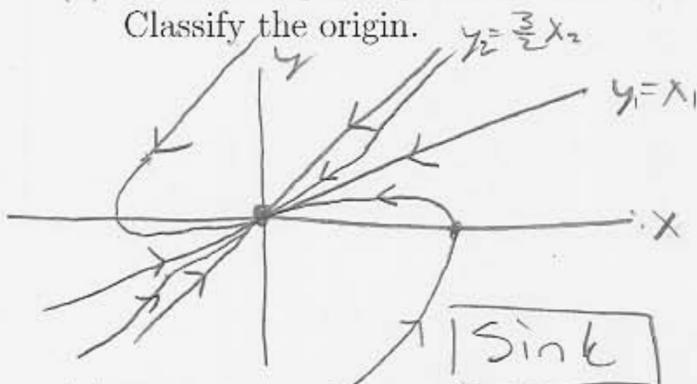
$$\begin{pmatrix} \alpha \\ \alpha \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Let $\alpha = 1$

$$\vec{y}(t) = k_1 e^{-2t} \begin{pmatrix} 2 \\ 3 \end{pmatrix} + k_2 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

(b) Sketch the phase portrait. Make sure you include the straight-line solutions and at least two other trajectories.

Classify the origin.



5 straight-line solutions

$$\lambda_1 = -2: y_1 = \frac{3x_1}{2}$$

$$\vec{y}_1(t) = k_1 e^{-2t} \begin{pmatrix} 2 \\ 3 \end{pmatrix} \rightarrow \vec{0} \text{ as } t \rightarrow \infty$$

$$\lambda_2 = -1: y_2 = x_2$$

$$\vec{y}_2(t) = k_2 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rightarrow \vec{0} \text{ as } t \rightarrow \infty$$

General soln

$$\vec{y}(t) = k_1 e^{-2t} \begin{pmatrix} 2 \\ 3 \end{pmatrix} + k_2 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Dominates as $t \rightarrow \infty$ Dominates as $t \rightarrow \infty$

(c) Find the particular solution that satisfies the initial condition $(x(0), y(0)) = (1, 0)$. Report your solution as one real vector.

$$\vec{y}(0) = k_1 \begin{pmatrix} 2 \\ 3 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2k_1 + k_2 \\ 3k_1 + k_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{aligned} 2k_1 + k_2 &= 1 & 3k_1 + k_2 &= 0 \\ 2k_1 - 3k_1 &= 1 & k_2 &= -3k_1 \\ -k_1 &= 1 & k_2 &= 3 \\ k_1 &= -1 & & \end{aligned}$$

$$\vec{y}(t) = -e^{-2t} \begin{pmatrix} 2 \\ 3 \end{pmatrix} + 3e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \vec{y}(t) = \begin{pmatrix} -2e^{-2t} + 3e^{-t} \\ -3e^{-2t} + 3e^{-t} \end{pmatrix}$$

3. Given the system

$$Y' = \begin{pmatrix} 2 & 8 \\ -1 & -2 \end{pmatrix} Y \quad A - \lambda I = \begin{pmatrix} 2-\lambda & 8 \\ -1 & -2-\lambda \end{pmatrix}$$

(a) Find the general solution.

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ (2-\lambda)(-2-\lambda) + 8 &= 0 \\ \lambda^2 + 4 &= 0 \\ \lambda^2 &= -4 \\ \lambda &= \pm 2i \\ \lambda_1 &= 2i, \lambda_2 = -2i \end{aligned}$$

$$\lambda_1 = 2i: (A - \lambda_1 I) \vec{v}_1 = \vec{0}$$

$$\begin{pmatrix} 2-2i & 8 \\ -1 & -2-2i \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(2-2i)x_1 + 8y_1 = 0$$

$$y_1 = \left(\frac{1}{4} + \frac{1}{4}i\right)x_1, \quad x_1 = \alpha$$

$$\begin{pmatrix} \alpha \\ \left(\frac{1}{4} + \frac{1}{4}i\right)\alpha \end{pmatrix}, \quad \alpha = 4, \quad \vec{v}_1 = \begin{pmatrix} 4 \\ 1+i \end{pmatrix}$$

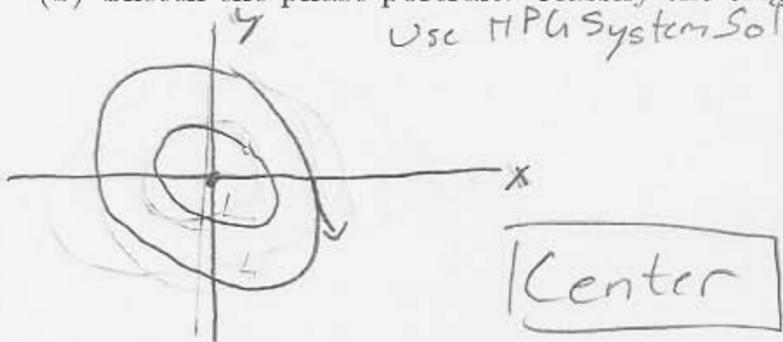
$$e^{2it} \begin{pmatrix} 4 \\ 1+i \end{pmatrix} = (\cos(2t) + i\sin(2t)) \begin{pmatrix} 4 \\ 1+i \end{pmatrix}$$

$$= \begin{pmatrix} 4\cos(2t) \\ -\cos(2t) - \sin(2t) \end{pmatrix} + i \begin{pmatrix} 4\sin(2t) \\ \cos(2t) - \sin(2t) \end{pmatrix}$$

$$\vec{y}(t) = k_1 \begin{pmatrix} 4\cos(2t) \\ \cos(2t) - \sin(2t) \end{pmatrix} + k_2 \begin{pmatrix} 4\sin(2t) \\ \cos(2t) + \sin(2t) \end{pmatrix}$$

(b) Sketch the phase portrait. Classify the origin.

Use HPK System Solver.



$$\lambda = \pm 2i = a \pm bi$$

$$a = 0, \quad b = 2$$

$$\vec{y} = (1, 0): A\vec{y} = \begin{pmatrix} 2 & 8 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

or See Attached

(c) Find the particular solution that satisfies the initial condition $Y(0) = (8, 6)$. Report your solution as one real vector.

$$\vec{y}(0) = k_1 \begin{pmatrix} 4 \\ 1+i \end{pmatrix} + k_2 \begin{pmatrix} 4 \\ 1-i \end{pmatrix} = \begin{pmatrix} 4k_1 \\ -k_1 + k_2 \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \end{pmatrix}$$

$$\begin{aligned} 4k_1 &= 8 & -k_1 + k_2 &= 6 \\ k_1 &= 2 & k_2 &= 8 \end{aligned}$$

$$\vec{y}(t) = 2 \begin{pmatrix} 4\cos(2t) \\ -\cos(2t) - \sin(2t) \end{pmatrix} + 8 \begin{pmatrix} 4\sin(2t) \\ \cos(2t) + \sin(2t) \end{pmatrix}, \quad \vec{y}(t) = \begin{pmatrix} 8\cos(2t) + 32\sin(2t) \\ -6\cos(2t) + 10\sin(2t) \end{pmatrix}$$

4. Given the second-order differential equation

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} + y = 0$$

(a) Find the corresponding first-order system by letting $v = \frac{dy}{dt}$.

$$v = \frac{dy}{dt}$$

$$\frac{dv}{dt} = \frac{d^2y}{dt^2} = -y - \frac{dy}{dt} = -y - v$$

$$\begin{cases} \frac{dy}{dt} = v \\ \frac{dv}{dt} = -y - v \end{cases} \quad \text{or } \frac{d\vec{y}}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \vec{y}$$

$$\vec{y} = \begin{pmatrix} y \\ v \end{pmatrix}$$

(b) Find the general solution.

$$\det(A - \lambda I) = \det \begin{pmatrix} -\lambda & 1 \\ -1 & -1-\lambda \end{pmatrix} = 0$$

$$-\lambda(-1-\lambda) + 1 = 0$$

$$\lambda^2 + \lambda + 1 = 0$$

$$\lambda = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$\lambda = \frac{-1 \pm \sqrt{3}i}{2}$$

$$\lambda = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

$$e^{(-\frac{1}{2} + \frac{\sqrt{3}}{2}i)t} \begin{pmatrix} 2 \\ -1 + \sqrt{3}i \end{pmatrix} = e^{-\frac{1}{2}t} (\cos(\frac{\sqrt{3}}{2}t) + i \sin(\frac{\sqrt{3}}{2}t)) \begin{pmatrix} 2 \\ -1 + \sqrt{3}i \end{pmatrix}$$

$$\lambda_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \quad (A - \lambda_1 I) \vec{v}_1 = \vec{0}$$

$$\begin{pmatrix} \frac{1}{2} - \frac{\sqrt{3}}{2}i & 1 \\ -1 & -\frac{1}{2} - \frac{\sqrt{3}}{2}i \end{pmatrix} \begin{pmatrix} v_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(\frac{1}{2} - \frac{\sqrt{3}}{2}i)y_1 + v_1 = 0$$

$$v_1 = (-\frac{1}{2} + \frac{\sqrt{3}}{2}i)y_1$$

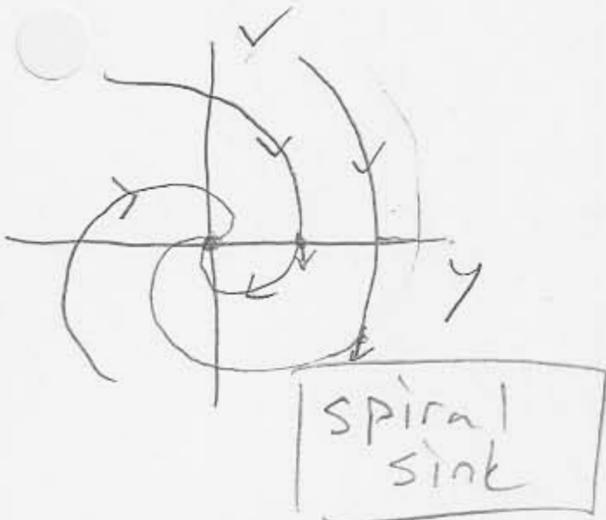
$$y_1 = \alpha$$

$$\begin{pmatrix} \alpha \\ (-\frac{1}{2} + \frac{\sqrt{3}}{2}i)\alpha \end{pmatrix}, \lambda = 2, \vec{v}_1 = \begin{pmatrix} 2 \\ -1 + \sqrt{3}i \end{pmatrix}$$

$$e^{-\frac{1}{2}t} \begin{pmatrix} 2 \cos(\frac{\sqrt{3}}{2}t) \\ -\cos(\frac{\sqrt{3}}{2}t) - \sqrt{3} \sin(\frac{\sqrt{3}}{2}t) \end{pmatrix} + i e^{-\frac{1}{2}t} \begin{pmatrix} 2 \sin(\frac{\sqrt{3}}{2}t) \\ -\sin(\frac{\sqrt{3}}{2}t) + \sqrt{3} \cos(\frac{\sqrt{3}}{2}t) \end{pmatrix}$$

$$\vec{y}(t) = k_1 e^{-\frac{1}{2}t} \begin{pmatrix} 2 \cos(\frac{\sqrt{3}}{2}t) \\ -\cos(\frac{\sqrt{3}}{2}t) - \sqrt{3} \sin(\frac{\sqrt{3}}{2}t) \end{pmatrix} + k_2 e^{-\frac{1}{2}t} \begin{pmatrix} 2 \sin(\frac{\sqrt{3}}{2}t) \\ \sqrt{3} \cos(\frac{\sqrt{3}}{2}t) - \sin(\frac{\sqrt{3}}{2}t) \end{pmatrix}$$

(c) Sketch the phase portrait. Classify the origin.



$$\lambda = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

$$= a \pm bi$$

$$a = -\frac{1}{2}, b = \frac{\sqrt{3}}{2}$$

$$\vec{y} = (1, 0)$$

$$A\vec{y} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\vec{y} = (2, -2): A\vec{y} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$

(d) What is the natural period of the system?

$$\frac{2\pi}{b} = \frac{2\pi}{\frac{\sqrt{3}}{2}} = \frac{4\pi}{\sqrt{3}}$$

(e) Find the particular solution that satisfies the initial condition $\vec{y}(0) = (2, -2)$. Report your solution as one real vector.

$$\vec{y}(0) = k_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix} = \begin{pmatrix} 2k_1 \\ -k_1 + \sqrt{3}k_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

$$2k_1 = 2 \quad -k_1 + \sqrt{3}k_2 = -2$$

$$k_1 = 1 \quad \sqrt{3}k_2 = -1$$

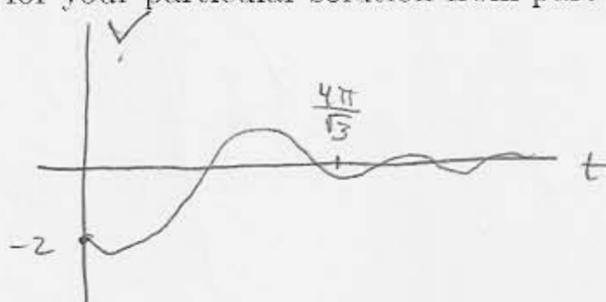
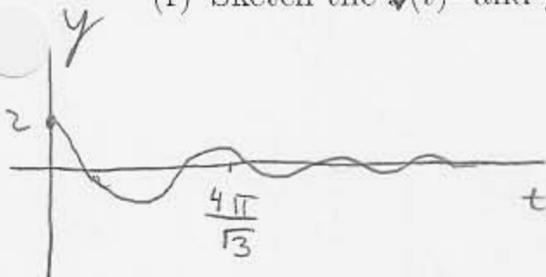
$$k_2 = -\frac{1}{\sqrt{3}}$$

$$= -\frac{\sqrt{3}}{3}$$

$$\vec{y}(t) = e^{-\frac{1}{2}t} \begin{pmatrix} 2 \cos(\frac{\sqrt{3}}{2}t) \\ -\cos(\frac{\sqrt{3}}{2}t) - \sqrt{3} \sin(\frac{\sqrt{3}}{2}t) \end{pmatrix} - \frac{\sqrt{3}}{3} e^{-\frac{1}{2}t} \begin{pmatrix} 2 \sin(\frac{\sqrt{3}}{2}t) \\ \sqrt{3} \cos(\frac{\sqrt{3}}{2}t) - \sin(\frac{\sqrt{3}}{2}t) \end{pmatrix}$$

$$\vec{y}(t) = \begin{pmatrix} 2e^{-\frac{1}{2}t} \cos(\frac{\sqrt{3}}{2}t) - \frac{2\sqrt{3}}{3} e^{-\frac{1}{2}t} \sin(\frac{\sqrt{3}}{2}t) \\ -2e^{-\frac{1}{2}t} \cos(\frac{\sqrt{3}}{2}t) - \frac{2\sqrt{3}}{3} e^{-\frac{1}{2}t} \sin(\frac{\sqrt{3}}{2}t) \end{pmatrix}$$

(f) Sketch the $v(t)$ - and $y(t)$ -graphs for your particular solution from part (e).



4. (Alternative Solution)

$$\frac{d^2 y}{dt^2} + \frac{dy}{dt} + y = 0$$

(a) $\frac{dx}{dt} = -x - y$ $\frac{dy}{dt} = x$ $\underline{\text{or}} \frac{d\vec{Y}}{dt} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \vec{Y}, \vec{Y} = \begin{pmatrix} x \\ y \end{pmatrix}$

(b) $A - \lambda I = \begin{pmatrix} -1-\lambda & -1 \\ 1 & -\lambda \end{pmatrix}$

$$\det \begin{pmatrix} -1-\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = 0$$

$$(-1-\lambda)(-\lambda) + 1 = 0$$

$$\lambda^2 + \lambda + 1 = 0$$

$$\lambda = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

$$\lambda_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \therefore (A - \lambda_1 I) \vec{v}_1 = \vec{0}$$

$$\begin{pmatrix} -\frac{1}{2} - \frac{\sqrt{3}}{2}i & -1 \\ 1 & \frac{1}{2} - \frac{\sqrt{3}}{2}i \end{pmatrix} \begin{pmatrix} v_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)v_1 - y_1 = 0$$

$$y_1 = \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)v_1$$

$$v_1 = \alpha, y_1 = \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\alpha$$

$$\begin{pmatrix} \alpha \\ \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\alpha \end{pmatrix} \text{ let } \alpha = 2$$

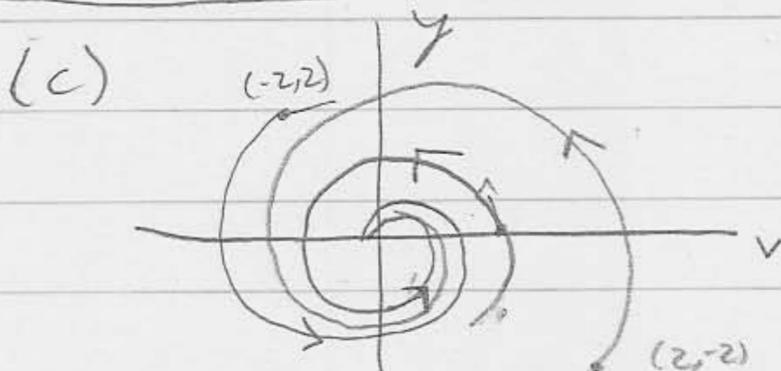
$$\vec{v}_1 = \begin{pmatrix} 2 \\ -1 - \sqrt{3}i \end{pmatrix}$$

$$e^{\lambda_1 t} \vec{v}_1 = e^{\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)t} \begin{pmatrix} 2 \\ -1 - \sqrt{3}i \end{pmatrix} = e^{-\frac{1}{2}t} e^{\frac{\sqrt{3}}{2}it} \begin{pmatrix} 2 \\ -1 - \sqrt{3}i \end{pmatrix}$$

$$= e^{-\frac{1}{2}t} \left(\cos\left(\frac{\sqrt{3}}{2}t\right) + i \sin\left(\frac{\sqrt{3}}{2}t\right) \right) \begin{pmatrix} 2 \\ -1 - \sqrt{3}i \end{pmatrix}$$

$$= e^{-\frac{1}{2}t} \left(\begin{matrix} 2\cos\left(\frac{\sqrt{3}}{2}t\right) \\ -\cos\left(\frac{\sqrt{3}}{2}t\right) + \sqrt{3}\sin\left(\frac{\sqrt{3}}{2}t\right) \end{matrix} \right) + i e^{-\frac{1}{2}t} \left(\begin{matrix} 2\sin\left(\frac{\sqrt{3}}{2}t\right) \\ -\sqrt{3}\cos\left(\frac{\sqrt{3}}{2}t\right) - \sin\left(\frac{\sqrt{3}}{2}t\right) \end{matrix} \right)$$

$$\vec{Y}(t) = k_1 e^{-\frac{1}{2}t} \begin{pmatrix} 2\cos\left(\frac{\sqrt{3}}{2}t\right) \\ -\cos\left(\frac{\sqrt{3}}{2}t\right) + \sqrt{3}\sin\left(\frac{\sqrt{3}}{2}t\right) \end{pmatrix} + k_2 e^{-\frac{1}{2}t} \begin{pmatrix} 2\sin\left(\frac{\sqrt{3}}{2}t\right) \\ -\sqrt{3}\cos\left(\frac{\sqrt{3}}{2}t\right) - \sin\left(\frac{\sqrt{3}}{2}t\right) \end{pmatrix}$$



$$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Spiral Sink

4. (Alternative Solution)

$$(d) \frac{2\pi}{b} = \frac{2\pi}{\frac{\sqrt{3}}{2}} = \boxed{\frac{4\pi}{\sqrt{3}}}$$

$$(e) \vec{y}(0) = k_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ -\sqrt{3} \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} = \begin{pmatrix} v(0) \\ y(0) \end{pmatrix}$$

$$2k_1 = -2$$

$$k_1 = -1$$

$$-k_1 - \sqrt{3}k_2 = 2$$

$$-\sqrt{3}k_2 = 1$$

$$k_2 = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

$$\vec{y}(t) = e^{-\frac{1}{2}t} \begin{pmatrix} -2 \cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{2}{\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2}t\right) \\ 2 \cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{2\sqrt{3}}{3} \sin\left(\frac{\sqrt{3}}{2}t\right) \end{pmatrix}$$

↑ previous solution flipped
↓

(f) Same as before

OR

Since not specified

$$\vec{y}(0) = k_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ -\sqrt{3} \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} v(0) \\ y(0) \end{pmatrix}$$

$$2k_1 = 2$$

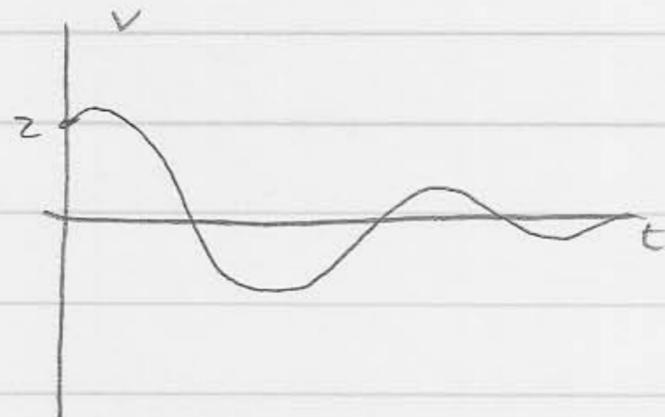
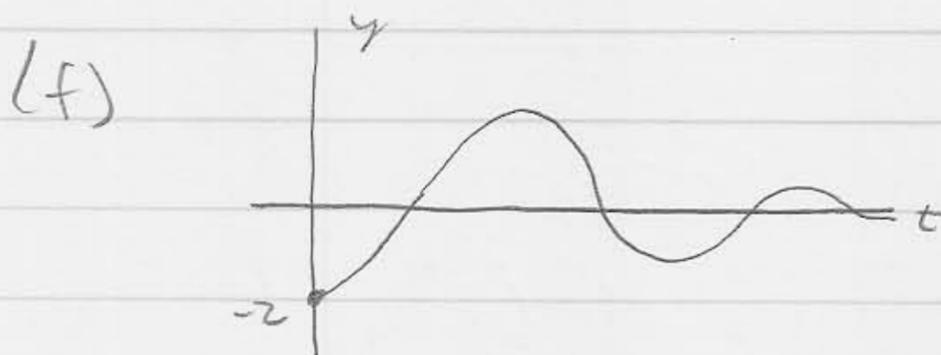
$$k_1 = 1$$

$$-k_1 - \sqrt{3}k_2 = -2$$

$$-\sqrt{3}k_2 = -1$$

$$k_2 = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\vec{y}(t) = e^{-\frac{1}{2}t} \begin{pmatrix} 2 \cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{2\sqrt{3}}{3} \sin\left(\frac{\sqrt{3}}{2}t\right) \\ -2 \cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{2\sqrt{3}}{3} \sin\left(\frac{\sqrt{3}}{2}t\right) \end{pmatrix} \quad \begin{matrix} * \text{Different} \\ \text{IC} \end{matrix}$$



5. Given the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -1 & 3 \\ -3 & 5 \end{pmatrix} \mathbf{Y} \quad A - \lambda I = \begin{pmatrix} -1-\lambda & 3 \\ -3 & 5-\lambda \end{pmatrix}$$

(a) Find the particular solution with the initial condition $\mathbf{Y}(0) = (1, 2)$. Report your solution as 1 real vector.

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ \begin{pmatrix} -1-\lambda & 3 \\ -3 & 5-\lambda \end{pmatrix} &= 0 \\ (-1-\lambda)(5-\lambda) + 9 &= 0 \\ \lambda^2 - 4\lambda + 4 &= 0 \\ (\lambda - 2)^2 &= 0 \\ \lambda &= \lambda_1 = \lambda_2 = 2 \end{aligned}$$

$$\begin{aligned} \vec{Y}(t) &= e^{\lambda t} \vec{V}_0 + t e^{\lambda t} \vec{V}_1 \\ \vec{V}_0 &= \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \quad \vec{V}_1 = (A - \lambda I) \vec{V}_0 \\ \vec{V}_0 &= \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \vec{V}_1 = \begin{pmatrix} -3 & 3 \\ -3 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \\ \vec{Y}(t) &= e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + t e^{2t} \begin{pmatrix} 3 \\ 3 \end{pmatrix} \\ \vec{Y}(t) &= \begin{pmatrix} e^{2t} + 3te^{2t} \\ 2e^{2t} + 3te^{2t} \end{pmatrix} \end{aligned}$$

(b) Graph the phase portrait using HPGSystemSolver (from the software associated with your book). Graph at least three separate trajectories in the phase portrait. Print and include your results.

See Attached.

6. Given the system

$$\begin{aligned} \frac{dx}{dt} &= 3x + 6y \\ \frac{dy}{dt} &= -x - 2y \end{aligned} \quad A = \begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix}, \quad A - \lambda I = \begin{pmatrix} 3-\lambda & 6 \\ -1 & -2-\lambda \end{pmatrix}$$

(a) Find the general solution.

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ (3-\lambda)(-2-\lambda) + 6 &= 0 \\ \lambda^2 - \lambda &= 0 \\ \lambda(\lambda - 1) &= 0 \\ \lambda_1 &= 0, \lambda_2 = 1 \end{aligned}$$

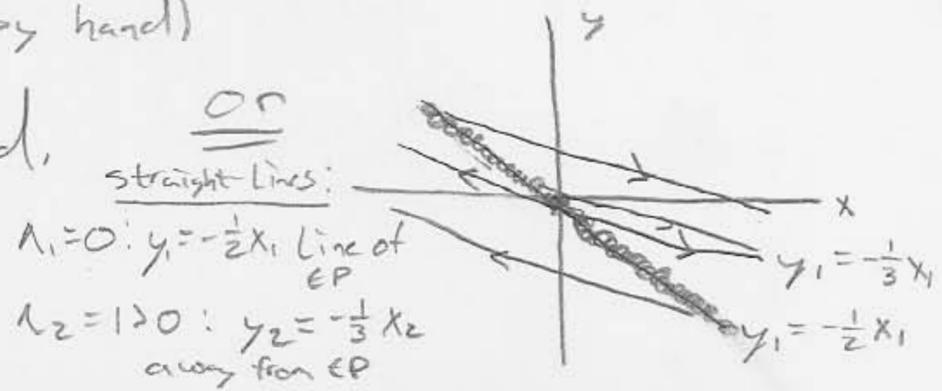
$$\begin{aligned} \lambda_1 = 0: (A - \lambda_1 I) \vec{V}_1 &= \vec{0} \\ \begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ 3x_1 + 6y_1 &= 0 \\ y_1 &= -\frac{1}{2}x_1, \quad x_1 = \alpha \\ \begin{pmatrix} \alpha \\ -\frac{\alpha}{2} \end{pmatrix}, \text{ let } \alpha = 2, & \\ \vec{V}_1 &= \begin{pmatrix} 2 \\ -1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \lambda_2 = 1: (A - \lambda_2 I) \vec{V}_2 &= \vec{0} \\ \begin{pmatrix} 2 & 6 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ 2x_2 + 6y_2 &= 0 \\ y_2 &= -\frac{1}{3}x_2, \quad x_2 = \alpha \\ \begin{pmatrix} \alpha \\ -\frac{\alpha}{3} \end{pmatrix}, \text{ let } \alpha = 3, & \\ \vec{V}_2 &= \begin{pmatrix} 3 \\ -1 \end{pmatrix} \end{aligned}$$

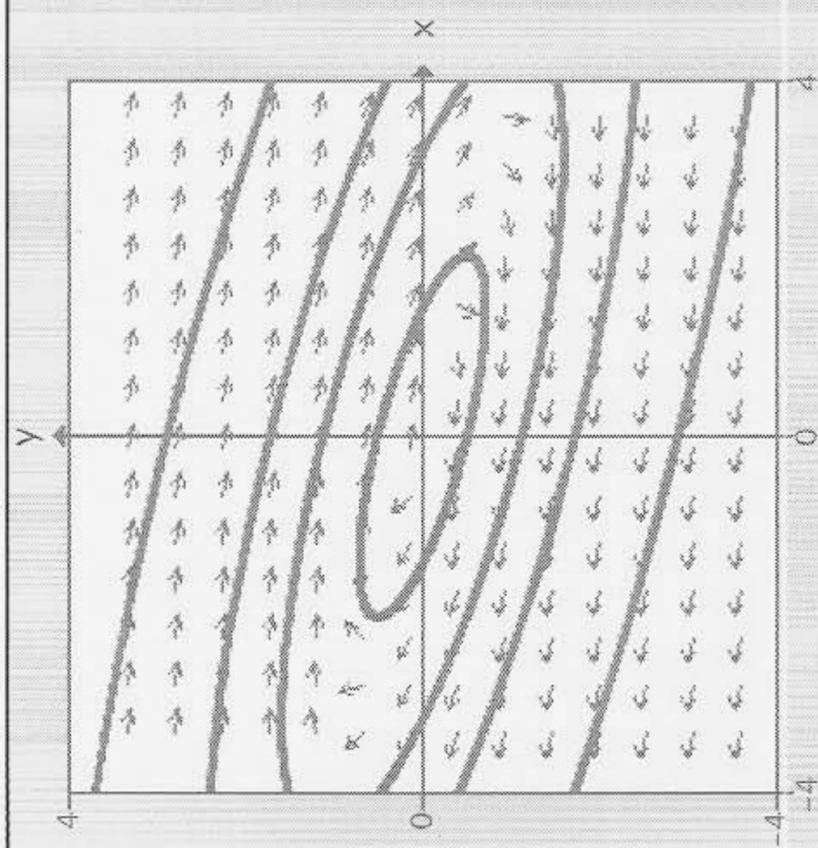
$$\vec{Y}(t) = k_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix} + k_2 e^t \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

(b) Graph the phase portrait using HPGSystemSolver. Graph at least three separate trajectories in the phase portrait. Print and include your results. (or sketch by hand)

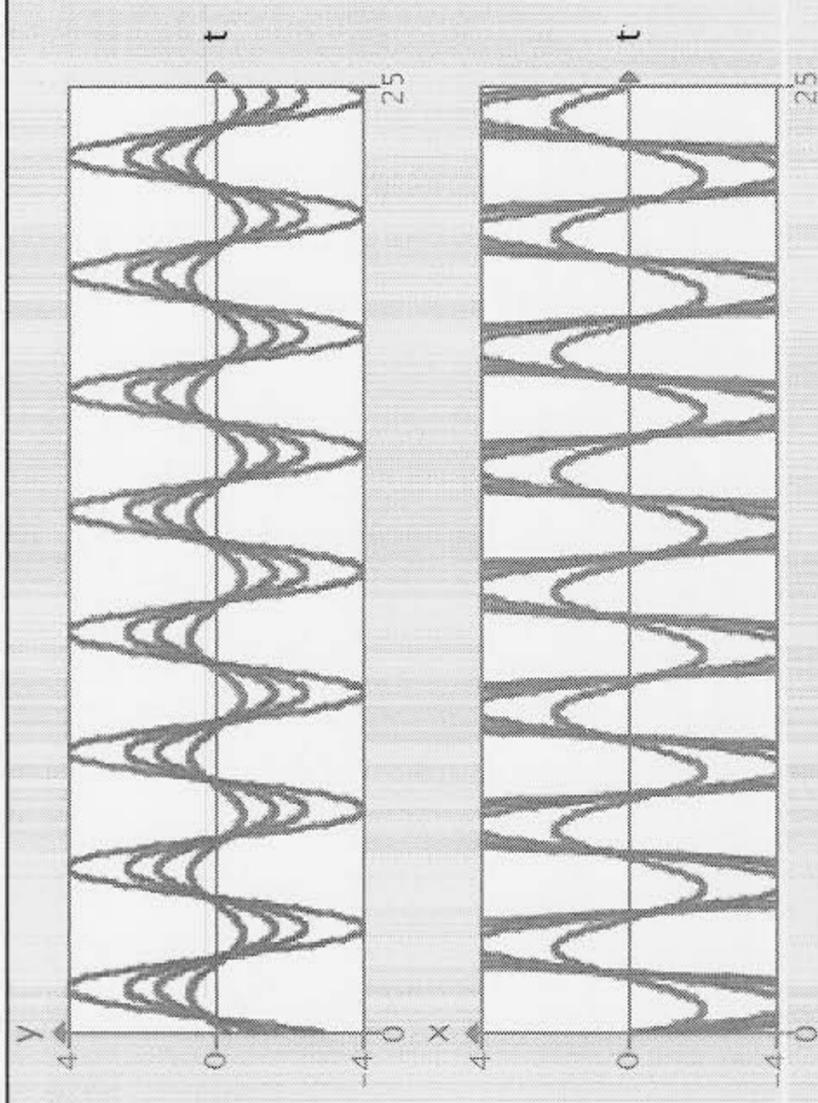
See Attached.



3.(b)



Clear Hide Field



Clear Overlay Time Graphs

Runge Kutta 4 Draw Solutions
 Draw Vectors

$dx/dt = 2*x+8*y$

$dy/dt = -x-2*y$

min x max x
 min y max y
 min t max t

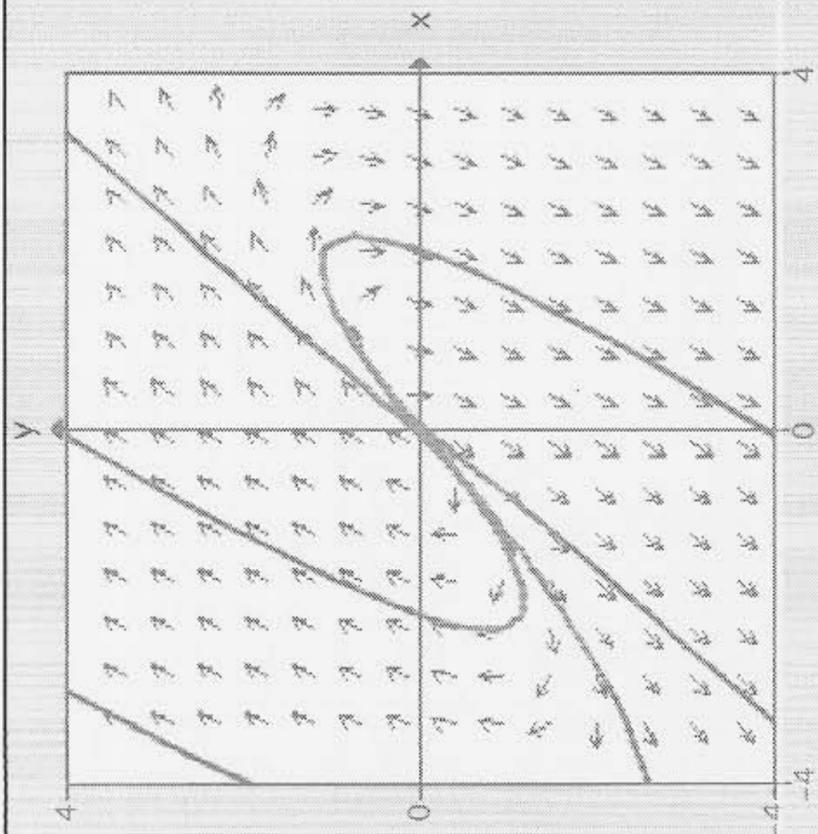
Reset Zoom Out Zoom In

x₀
 y₀
 t₀

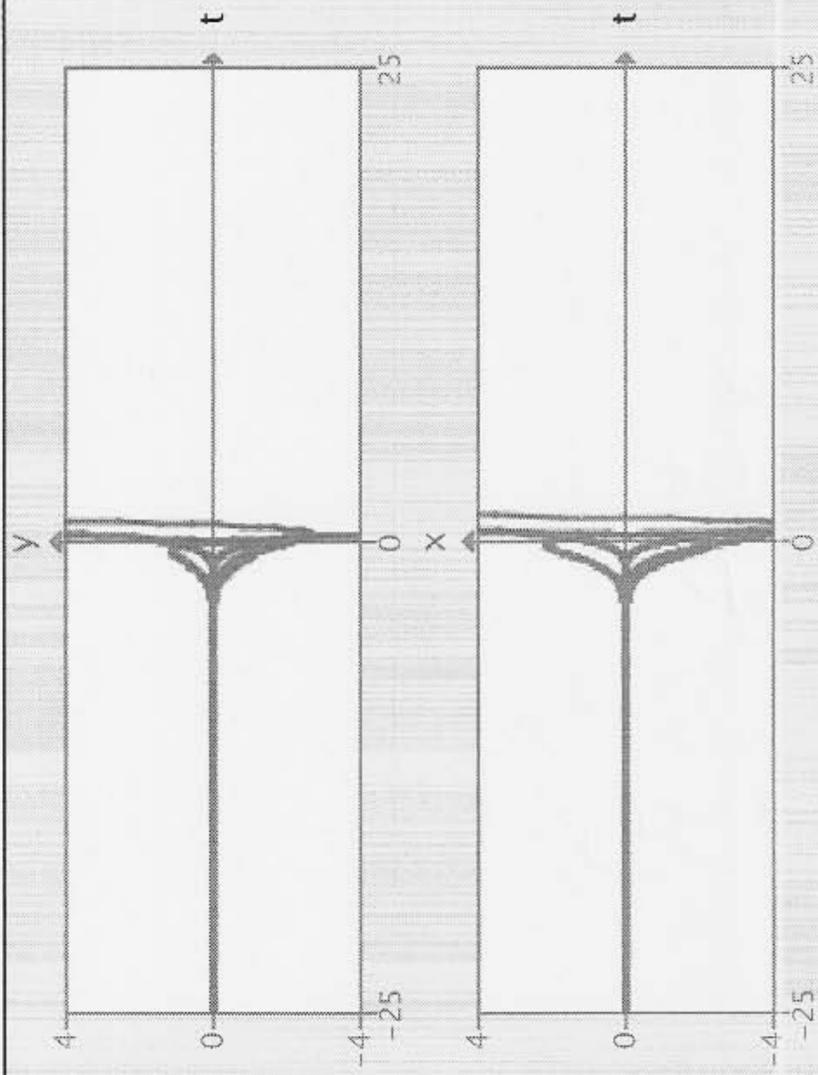
Solution

delta t Equations

5.(b)



Clear Hide Field



Clear Overlay Time Graphs

- Runge Kutta 4
- Draw Solutions
- Draw Vectors

$dx/dt = -x + 3*y$

$dy/dt = -3*x + 5*y$

min x max x
 min y max y
 min t max t

Reset Zoom Out Zoom In

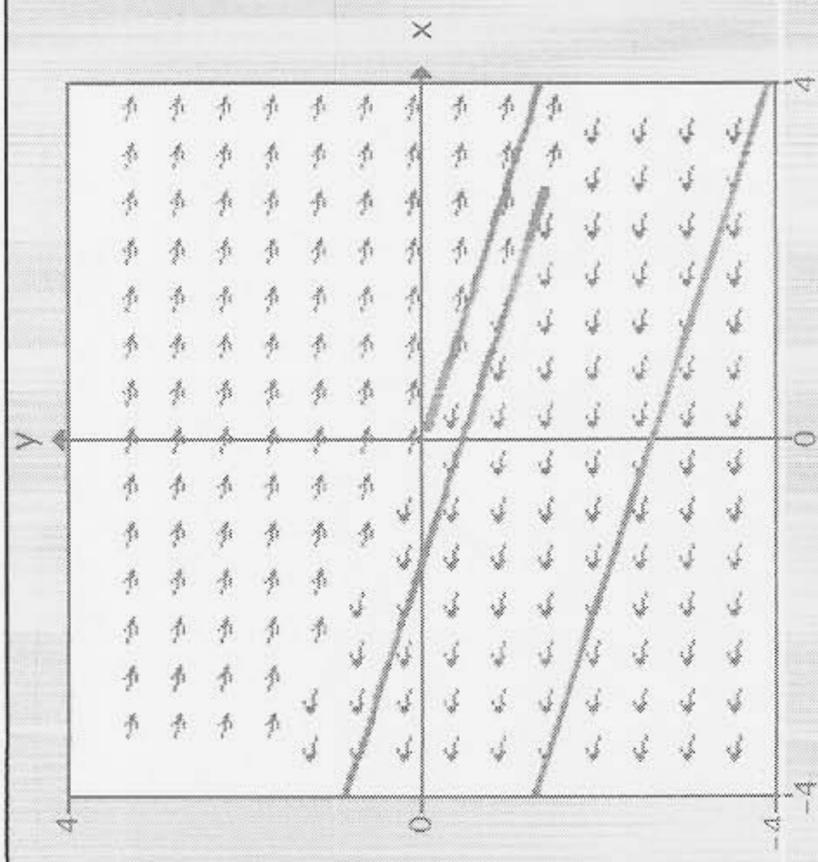
x₀
 y₀
 t₀

Solution

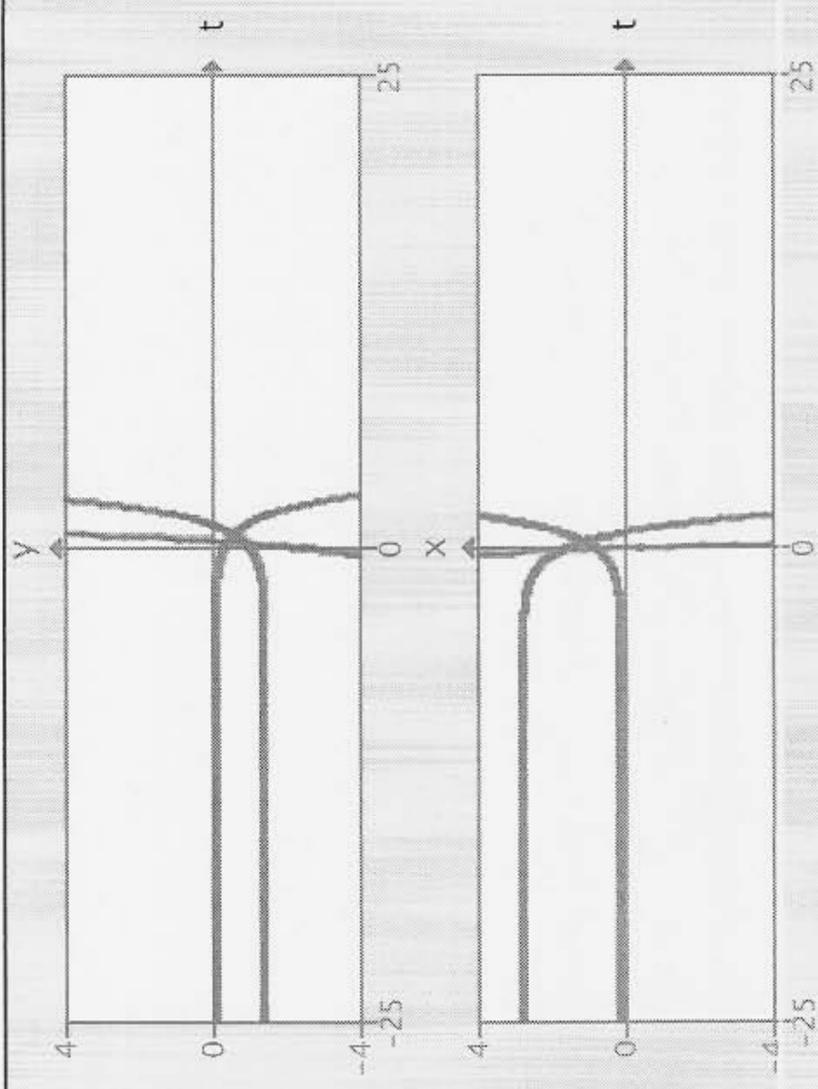
delta t

Equations

6.(b)



Clear Hide Field



Clear Overlay Time Graphs

Runge Kutta 4 Draw Solutions Draw Vectors

$dx/dt = 3*x+6*y$

$dy/dt = -x-2*y$

min x max x
 min y max y
 min t max t

Reset Zoom Out Zoom In

x₀
 y₀
 t₀

Solution

delta t

Equations