

## The Potential Formulation

$$\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \underline{\underline{\vec{B} = \vec{\nabla} \times \vec{A}}}$$

$$\vec{\nabla} \times \vec{E} = - \frac{\partial \vec{B}}{\partial t} \quad (\text{Faraday's Law})$$

$$= - \frac{\partial}{\partial t} \vec{\nabla} \times \vec{A} = - \vec{\nabla} \times \frac{\partial \vec{A}}{\partial t}$$

$$\vec{\nabla} \times \left( \vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0$$

$$\Rightarrow \vec{E} + \frac{\partial \vec{A}}{\partial t} = - \vec{\nabla} V$$

$$\vec{E} = - \vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$$

$$\underline{\underline{\vec{B} = \vec{\nabla} \times \vec{A}}}$$

$$\square^2 = \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \quad L \equiv \vec{\nabla} \cdot \vec{A} + \frac{1}{c^2} \frac{\partial V}{\partial t}$$

Put  $\vec{E}$  &  $\vec{B}$  into Faraday's Law & Ampere/Maxwell Law to

Show :

$$\square^2 V + \frac{\partial L}{\partial t} = - \frac{f}{\epsilon_0} \quad \text{and} \quad \square^2 \vec{A} - \vec{\nabla} L = - \mu_0 \vec{J}$$

### Gauge Transformations

Remember that  $\vec{E}, \vec{B}$  are the measurable quantities (not  $V, \vec{A}$ ).

We can change  $V, \vec{A}$  as long as we do not change  $\vec{E}, \vec{B}$ .

These are called Gauge Transformations.

$$\vec{A}' = \vec{A} + \vec{\nabla} \lambda \quad \text{where } \lambda \text{ is any scalar function}$$

$$V' = V - \frac{\partial \lambda}{\partial t}$$

$$\vec{B}' = \vec{B}, \quad \vec{E}' = \vec{E}$$

### Coulomb Gauge

In magnetostatics  $\vec{\nabla} \cdot \vec{A} = 0$

$$L = \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2}$$

$$\underbrace{\nabla^2 V - \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2}}_{\square^2 V} + \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon_0}$$

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \quad \text{Poisson Equation}$$

$$V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}', t')}{r} d\tau'$$

$$\square^2 \vec{A} = -\mu_0 \vec{j} + \frac{1}{c^2} \vec{\nabla} \left( \frac{\partial V}{\partial t} \right)$$

### Lorentz Gauge

$$L=0$$

$$\square^2 V = -\frac{\rho}{\epsilon_0} \quad \square^2 \vec{A} = -\mu_0 \vec{j}$$

## Retarded Potentials

### Lorentz Gauge

$$L=0$$

$$\square^2 V = -\frac{\rho}{\epsilon_0} \quad \square^2 \vec{A} = -\mu_0 \vec{J}$$

In the static case

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \quad \nabla^2 \vec{A} = -\mu_0 \vec{J}$$

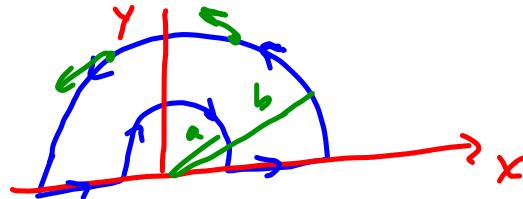
$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}') d\tau'}{r} \quad \vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}') d\tau'}{r}$$

retarded time accounts for the finite speed of light

$$t_r = t - \frac{r}{c}$$

$$V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}', t_r) d\tau'}{r} \rightarrow \text{retarded potentials}$$

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}', t_r) d\tau'}{r}$$



$$I(t) = kt \quad k = \text{constant}$$

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{S}(r', t_r)}{r'} d\tau'$$

$$t_r = t - \frac{R}{c}$$

Find  $\vec{A}$  at the origin.

$$\vec{A} = \frac{\mu_0}{4\pi} \int_{\text{loop}} \frac{I(t_r)}{r} d\vec{r} = \frac{\mu_0}{4\pi} \int_{\text{loop}} k \left( t - \frac{R}{c} \right) \frac{d\vec{r}}{r}$$

$$= \frac{\mu_0}{4\pi} k \left[ \int_{\text{loop}} \frac{t}{r} d\vec{r} - \frac{1}{c} \int_{\text{loop}} d\vec{r} \right]$$

$$= \frac{\mu_0 k t}{4\pi} \int_{\text{loop}} \frac{d\vec{r}}{r}$$

$$= \frac{\mu_0 k t}{4\pi} \left[ \frac{1}{b} (-\pi b \hat{x}) + \frac{1}{a} \pi a \hat{x} + 2 \hat{x} \int_a^b \frac{dx}{x} \right]$$

$$= \frac{\mu_0 k t}{4\pi} \left[ 2 \ln\left(\frac{b}{a}\right) \right] \hat{x}$$

Point charge in a constant  $E$ -field

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r} + E_0 \hat{z}$$
$$\vec{F} = q E_0 \hat{z}$$
$$T_{ij} = \epsilon_0 [E_i E_j - \frac{1}{2} \delta_{ij} E^2]$$

$$\vec{F} = \oint_{\text{boundary area}} \vec{T} \cdot d\vec{s} = \mu_0 \epsilon_0 \underbrace{\int_0^T \int_S \vec{s} \cdot d\tau}_{0}$$

Pick sphere of radius  $R$

By symmetry, we only have the  $z$  component to find

$$F_z = \int_{\text{sphere}} (\vec{T} \cdot d\vec{s})_z = \int_{\text{sphere}} T_{zx} dx + T_{zy} dy + T_{zz} dz$$

$$= \epsilon_0 \int_{\text{sphere}} (E_x E_z + E_y E_z + (E_z E_z - \frac{1}{2} E^2)) d\sigma_z$$

$$= \epsilon_0 \int_{\text{sphere}} E_z (\vec{E} \cdot d\vec{s}) - \frac{1}{2} E^2 d\sigma_z$$

$$d\vec{s} = R^2 \sin \theta 2\pi d\theta \hat{r}$$

$$d\sigma_z = R^2 \sin \theta \cos \theta 2\pi d\theta$$

$$= \epsilon_0 \int_0^\pi R^2 2\pi \left[ E_0 + \frac{q}{4\pi\epsilon_0 R^2} \cos \theta \right] \left[ \frac{1}{4\pi\epsilon_0 R^2} + E_0 \cos \theta \right] \sin \theta d\theta$$

$$= \epsilon_0 \int_0^\pi R^2 2\pi \frac{E_0}{4\pi\epsilon_0 R^2} \theta (1 + \cos^2 \theta) \sin \theta d\theta$$

$$= \frac{E_0 q}{2} \left[ -\cos \theta - \frac{\cos^3 \theta}{3} \right]_0^\pi$$

$$= \frac{E_0 q}{2} \left( 2 + \frac{2}{3} \right) = \frac{5}{3} E_0 q$$

If you go through the  $-\int \frac{1}{2} E^2 d\sigma_z \rightarrow -\frac{1}{3} E_0 q$