Note Title 10/23/2006

10/23/06

As with vectors

$$\vec{z} = c_1\vec{q}_1 + c_2\vec{q}_2 + \cdots c_N\vec{q}_N$$

$$c_i = (\vec{q}_i, \vec{z}) = \vec{q}_i \cdot \vec{z}$$

if $\|\vec{q}_i\| = 1$

otherwise

$$c_i = \frac{(\vec{q}_i, \vec{z})}{(\vec{q}_i, \vec{q}_i)}$$

proof:

$$(\vec{q}_i, z) = (\vec{q}_i, c_1\vec{q}_1) + (\vec{q}_i, c_2\vec{q}_2) + \cdots$$

$$= c_1 \underbrace{(\vec{q}_i, \vec{q}_1)} + c_2 \underbrace{(\vec{q}_i, \vec{q}_2)} + \cdots$$

All the dot products are zero except

$$c_i (\vec{q}_i, \vec{q}_i)$$

$$(\vec{q}_i, \vec{z}) = c_i (\vec{q}_i, \vec{q}_i)$$

$$\Rightarrow c_i = \frac{(\vec{q}_i, \vec{z})}{(\vec{q}_i, \vec{q}_i)}$$

if the $\vec{q}_i$ are normalized, the denominator is 1.

Now we extend this idea to functions:

If the basis function $g_i(x)$ are orthonormal then we expect:

$$\boxed{c_i \sim (f(x), g_i(x))}$$

Define a "dot" product for functions

Let $u(x)$, $v(x)$ be defined on $[-\ell, \ell]$,

$$(u, v) \equiv \int_{-\ell}^{\ell} u(x) v(x)\, dx$$

So if

$$f(x) = c_1 g_1(x) + c_2 g_2(x) + \cdots c_N g_N(x) \cdots$$

inner product aka "dot" product

then $c_i \sim (f, g_i)$

Many important examples of orthogonal function expansions

In due course you will see

Legendre polynomials

Bessel functions

Spherical Harmonics

Hermite polynomials

⋮

Trigonometric Polynom.

i.e. Fourier Series

Recall from Last time

Kronecker - Delta

$$\boxed{\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}}$$

e.g. $\sum_{i=1}^{100} \sin(ix/\ell)\,\delta_{i3} = \sin(3x/\ell)$

$$\varphi_k(x) \equiv \sin(k\pi x/\ell)$$

$$\psi_k(x) \equiv \cos(k\pi x/\ell)$$

$$\boxed{\begin{gathered}(\varphi_i, \varphi_j) = (\psi_i, \psi_j) = \ell\,\delta_{ij} \\ (\varphi_i, \psi_j) = 0\end{gathered}}$$

This will be a HW problem

So if

$$f(x) = \frac{a_0}{2} + \sum_{N=1}^{\infty} a_N \cos(N\pi x/\ell) + b_N \sin(N\pi x/\ell)$$

$$f(x) = \frac{a_0}{2} + \sum_{N=1}^{\infty} a_N \psi_N(x) + b_N \varphi_N(x)$$

$\Rightarrow$

A] $a_N = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\left(\frac{N\pi x}{\ell}\right)dx = \frac{1}{\ell}(f, \psi_N)$

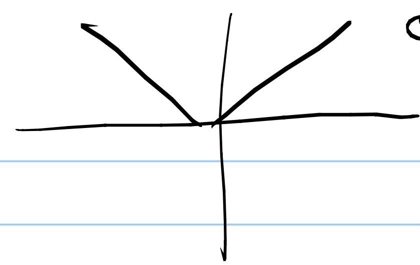
B] $b_N = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\left(\frac{N\pi x}{\ell}\right)dx = \frac{1}{\ell}(f, \varphi_N)$

or for the complex form

C) $$c_n = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)\, e^{-in\pi x/\ell}\, dx$$

Summary: to compute the Fourier Series of a function $f(x)$ you need to do integrals A, B, or C above.

Eg.



even function
cosines
only

$f(x) = |x|$ on $[-1, 1]$

$$a_0 = \int_{-1}^{1} f(x)\, dx$$

$$= 2\int_0^1 x\, dx = 2\,\frac{x^2}{2}\Big|_0^1$$

$$= 1$$

$$a_1 = \int_{-1}^{1} f(x)\cos(\pi x)\, dx$$

$$= 2\int_0^1 x\cos(\pi x)\, dx$$

$$= -4/\pi^2$$

$$a_2 = \int_{-1}^{1} |x| \cos(2\pi x)\, dx$$

$$= 0$$

$$a_3 = \int_{-1}^{1} |x| \cos(3\pi x)\, dx$$

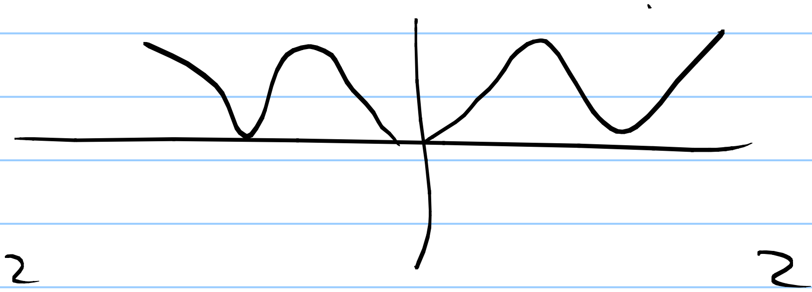
$$= -\frac{4}{9\pi^2}$$

$$\vdots$$

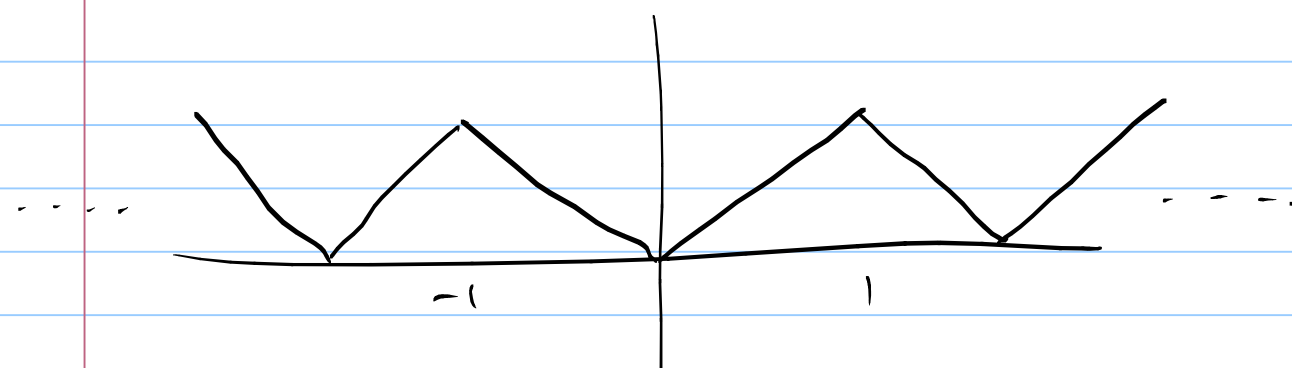
$$f(x) = \frac{a_0}{2} + a_1 \cos(\pi x) + a_3 \cos(3\pi x) \cdots$$

$$= \frac{1}{2} - \frac{4}{\pi^2} \cos(\pi x) - \frac{4}{9\pi^2} \cos(3\pi x)$$

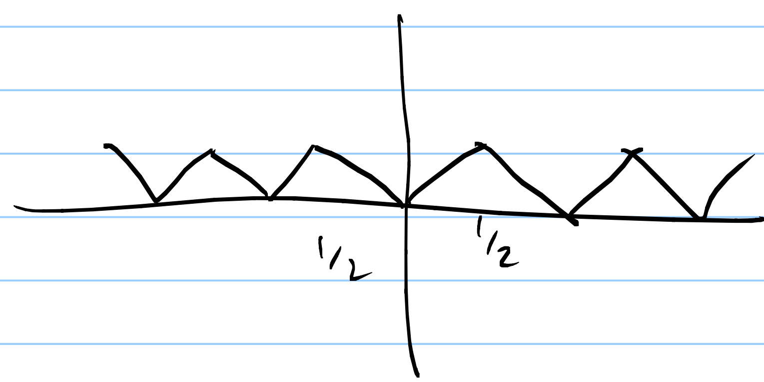
$$= \frac{1}{2} - \frac{4}{\pi^2}\cos(\pi x) - \frac{4}{9\pi^2}\cos(3\pi x)$$



NIntegrate[ ]



Periodic ext. of $|x|$ on $[-1,1]$



$$\sum_{N=-\infty}^{\infty} c_N \, e^{iN\pi x/\ell}$$ Fourier series

$$\downarrow$$

$$\lim_{\ell \to \infty}$$

$$\int c(k) \, e^{ikx} \, dk$$ Fourier integral