MACS 332 - July 20, 2006
Exam II - 50 minutes

NAME: Key.

In order to receive full credit, SHOW ALL YOUR WORK. Full credit will be given only if all reasoning and work is provided. Please enclose your final answers in boxes.

1. (10 points)
   Given that,

$$\mathbf{A} = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & -2 & 0 \\ 2 & 4 & -1 \\ 1 & 5 & 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 5 & 0 \\ 4 & 8 & -2 \\ 0 & -2 & 0 \end{bmatrix}.$$

   Calculate $\det(\mathbf{ABC})$.

$$\det(ABC) = \det(A)\det(B)\det(C) = \det(A)\cdot(-\det(A))\cdot 2\det(A) = -2\{\det(A)\}^3$$

$$\det(A) = 1\cdot\det\begin{pmatrix} 4 & -1 \\ -2 & 0 \end{pmatrix} - 5\det\begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix} =$$

$$= 1(-2) - 5\cdot 0 = -2 \quad \Rightarrow \det(ABC) = -2(-2)^3 = -2\cdot -8 = 16$$

2. (10 points) Given,

$$\mathbf{A} = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}.$$

   If $a^{-1}_{i,j}$ corresponds the the i, j element of $\mathbf{A}^{-1}$, calculate $a^{-1}_{2,3}$.

$$a^{-1}_{2,3} = \frac{1}{\det(A)}\cdot C_{32} = \frac{-1}{2}\cdot(-1)^{3+2}\det\begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} = +\frac{1}{2}(-1) = -\frac{1}{2}.$$

3. (10 points)
Assume that for $\mathbf{A}_{n\times n}$ the system of equations $\mathbf{Ax}=\mathbf{b}$ has infinitely many solutions for some $\mathbf{b}\in\mathbb{R}^n$.

Briefly respond to the following statements:

a. What is the determinant of **A**?

$\det(A) = 0$

b. What is the determinant of $\mathbf{A}^T$?

$\det(A) = 0$

c. What are the allowed dimensions of the null space of **A** .

$0 < \dim(\text{Nul}\,A) \leq n$

d. What are the allowed dimensions of the column space of **A**.

$0 \leq \dim(\text{Col}\,A) < n$

4. (10 points)

a. Let V be the set of all vectors in $\mathbb{R}^4$ of the form $\begin{bmatrix} 4a+3b \\ 0 \\ c-2 \\ a+b \end{bmatrix}$, where $a, b, c \in \mathbb{R}$. Is V a vector space?

Justify your answer. Since $V \subset \mathbb{R}^4$ we only need to show that V is closed under + and $\cdot$.

Let $u, v \in V$, $d \in \mathbb{R}$, then,

$$u + dv = \begin{bmatrix} 4a+3b \\ 0 \\ c-2 \\ a+b \end{bmatrix} + d\begin{bmatrix} 4a'+3b' \\ 0 \\ c'-2 \\ a'+b' \end{bmatrix} = \begin{bmatrix} 4(a+da')+3(b+db') \\ 0 \\ c+dc'-4 \\ da'+a+b+db' \end{bmatrix} =$$

$$= \begin{bmatrix} 4x+3y \\ 0 \\ z-2 \\ x+y \end{bmatrix} \in V$$

Since $x, y, z \in \mathbb{R}$

$\Rightarrow$ Yes. V is a Vector Space.

b. Let H be the subset of vectors from $\mathbb{R}^3$ of the form $\begin{bmatrix} 3t \\ 0 \\ -3t \end{bmatrix}$, where $t \in \mathbb{R}$.

Show that H is a subspace of $\mathbb{R}^3$.

5. (10 points) Given,

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 3 & 12 & 18 & 7 \\ 5 & 20 & 28 & 11 \end{bmatrix}.$$

a. Determine a basis for Row **A**.

b. Determine a basis for Col **A**.

c. What is the dimension of the null space of **A**? dim(Nul A) + dim(Col A) = 4 <=>

dim(Nul A) = 4 - 2 = 2.