Dispersive linear pulse propagation

start with $E(z,t) = A(z,t)\,e^{i(k_0 z - \omega_0 t)} + c.c.$

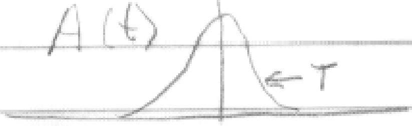
at $z=0$, $A(0,t)$ is a pulse, e.g. $A_0 e^{-t^2/T^2}$

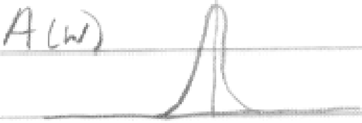
pulse has a finite bandwidth:

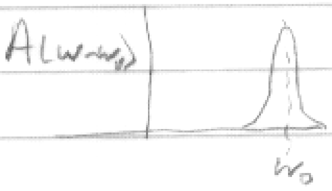
$$E(z,\omega) = \mathcal{F}\{E(z,t)\} = \int A(z,t)\,e^{i(k_0 z - \omega_0 t)}\,e^{i\omega t}\,dt$$

$$= A(z,\omega-\omega_0)\,e^{ik_0 z}$$

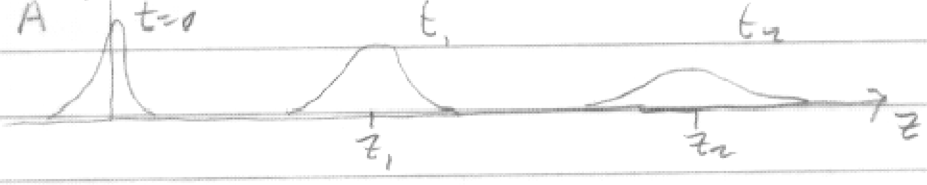
so schematically,





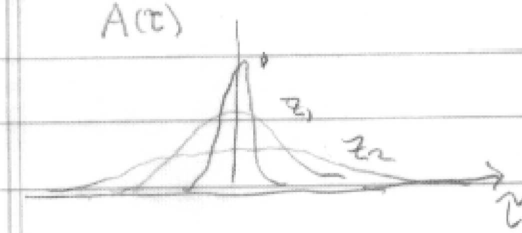


We expect this pulse to propagate forward and broaden:



It will be useful to separate out the changes in shape from the displacement in position → moving coord system.

→ $A(z,\tau)$ with $\tau = t - z/v_g$



Note: leading edge in space is at $\tau < 0$

wave equation in time domain:

$$-\nabla^2 E + \frac{1}{c^2}\frac{\partial^2 D}{\partial t^2} = \frac{4\pi}{c^2}\frac{\partial^2}{\partial t^2} P_{NL}$$

for now: 1D $\nabla^2 \rightarrow \partial_z^2$

linear $P_{NL} \rightarrow 0$

Note $D(z,\omega) = \varepsilon(\omega) E(z,\omega)$

so wave eqn. is easiest in freq. domain.

- substitute

$$E(z,t) = \frac{1}{2\pi}\int E(z,\omega)\, e^{-i\omega t}\, d\omega$$

$$D(z,t) = \frac{1}{2\pi}\int D(z,\omega)\, e^{-i\omega t}\, d\omega$$

- do ∂_t^2 on $e^{-i\omega t}$
- drop integral

$$\partial_z^2 E(z,\omega) + \varepsilon(\omega)\frac{\omega^2}{c^2} E(z,\omega) = 0$$

direct solution:

$E(z,\omega) = E(\omega)\, e^{ik(\omega)}$

$k^2(\omega) = \frac{\varepsilon(\omega)\omega^2}{c^2}$

→ use $E(z,\omega) = A(z,\omega-\omega_0)\, e^{ik_0 z}$ → envelope solution

$$\partial_z^2 A + 2ik_0\, \partial_z A + (k^2(\omega) - k_0^2) A = 0$$

where $k^2(\omega) \equiv \varepsilon(\omega)\omega^2/c^2$

(Notation: in fiber optics $k(\omega) \rightarrow \beta(\omega)$ which includes modal dispersio

Approximate $k(\omega)$ with Taylor expansion:

$$k(\omega) = k_0 + k_1(\omega-\omega_0) + D \quad \text{w/ } D = \sum_{n=2}^{\infty} \frac{1}{n!} k_n (\omega-\omega_0)^n$$

$$k_1 = \left.\frac{\partial k}{\partial \omega}\right|_{\omega_0} = \frac{1}{v_g} \qquad k_0 = \underline{n(\omega_0)\omega_0} \over c$$

D accounts for high-order dispersion

$$k^2(\omega) = k_0^2 + 2k_0k_1(\omega-\omega_0) + k_1^2(\omega-\omega_0)^2 + 2k_0D + 2k_1(\omega-\omega_0)D + \underset{\text{small}}{D^2}$$

$$\partial_z^2 A + 2ik_0\partial_z A + \left(2k_0k_1(\omega-\omega_0) + 2k_0D + 2k_1(\omega-\omega_0)D + k_1^2(\omega-\omega_0)^2\right)A = 0$$

transform back to time domain: mult by $\frac{1}{2\pi}e^{-i\omega' t}$ and integrate.

$$\mathfrak{F}^{-1}\{\omega F(\omega)\} = i\frac{\partial}{\partial t}f(t) \qquad \omega' = \omega - \omega_0$$

recall A is $A(z, \omega-\omega_0)$

$$\rightarrow \left[\partial_z^2 + 2ik_0\left(\partial_z + k_1\partial_t\right) + 2ik_1\tilde{D}\frac{\partial}{\partial t} + 2k_0\hat{D} - k_1\partial_t^2\right]A(z,t) = 0$$

$$\tilde{D} \text{ is } \mathfrak{F}^{-1}\{D\} \rightarrow \sum_{n=2}^{\infty}\frac{1}{n}k_n\left(i\partial_t\right)^n$$

high-order dispersion in time domain.

Now we move to a frame centered on pulse

$$z' = z \qquad \tau = t - k_1 z = t - z/v_g$$

$$\partial_z = \frac{\partial z'}{\partial z}\partial_{z'} + \frac{\partial \tau}{\partial z}\partial_\tau = \partial_{z'} - k_1\partial_\tau \qquad \partial_t = \partial_\tau$$

$$\left[\partial_{z'}^2 + \cancel{k_1^2\partial_\tau^2} - 2k_1\partial_{z'}\partial_\tau + 2ik_0\partial_{z'} + 2ik_1\tilde{D}\partial_\tau + 2k_0\tilde{D} - \cancel{k_1\partial_\tau}\right]A = 0$$

Approximations:

slowly-varying envelope $\partial_{z'}^2 \to 0$

$$k_1 = 1/v_g \quad k_0 = \omega_0/v_{ph} \rightarrow k_1/k_0 = \frac{v_{ph}}{v_g}\frac{1}{\omega_0} = \frac{n_g}{n_0}\frac{1}{\omega_0}$$

$$\rightarrow \left[2ik_0\partial_{z'}\left(1 + \frac{n_g}{n_0}\frac{i}{\omega_0}\partial_\tau\right) + 2k_0\tilde{D}\left(1 + \frac{n_g}{n_0}\frac{1}{\omega_0}\partial_\tau\right)\right]A = 0$$

in the limit of no dispersion, $n_g = n_0$ $\tilde{D} \to 0$

$$\to \partial_{z'}\left(\underbrace{\left(1+\frac{i}{\omega_0}\partial_\tau\right)A}_{\text{indep. of } z}\right) = 0 \qquad \to A(z,\tau) = A(0,\tau)$$

typically $\frac{1}{\omega_0}\partial_\tau$ is small: for $\partial_\tau A \sim \frac{1}{T}A$

$\omega_0 T >> 1$ and we drop $\frac{1}{\omega_0}\partial_\tau$ terms

For dispersive propag: keep k_2

$$D \to \frac{1}{2}k_2(\omega-\omega_0)^2 \qquad \tilde{D} \Rightarrow -\frac{1}{2}k_2\frac{\partial^2}{\partial\tau^2}$$

$$\to \frac{\partial A}{\partial z'} + \frac{1}{2}ik_2\frac{\partial^2 A}{\partial\tau^2} = 0$$

(this doesn't seem to work correctly for gaussian pulses, where $T^2(z) = T^2(0)(1+z^2/z_0^2)$)

Nonlinear propagation

RHS to wave eqn: $-\frac{4\pi^2}{c^2}\partial_t^2 P_{NL}(z,t)$

let $P(z,t) = p(z,t)\, e^{i(k_0 z - \omega_0 t)} + c.c.$

for instantaneous 3rd order response:

$$p(z,t) = 3\chi^{(3)}|A(z,t)|^2 A(z,t)$$

$$\partial_t P = \left(-i\omega_0 p + \partial_t p\right) e^{i(k_0 z - \omega_0 t)} + c.c.$$

$$= \left[-i\omega_0\left(1 + \frac{i}{\omega_0}\partial_t\right) p\right] e^{i(k_0 z - \omega_0 t)} + c.c.$$

$$\partial_t^2 P = -\omega_0^2\left[\left(1 + \frac{i}{\omega_0}\partial_t\right)^2 p\right] e^{i(k_0 z - \omega_0 t)} + c.c.$$

when we transform to moving frame

$$\rightarrow \text{RHS} = -\frac{4\pi\omega_0^2}{c^2}\left(1 + \frac{i}{\omega_0}\partial_\tau\right)^2 p$$

Note that nonlinearity is easiest to work with in the time domain.

- simplest approx is $\frac{1}{\omega_0}\partial_t \rightarrow 0$

so driving term is just $-\frac{4\pi\omega_0^2}{c^2} p(z,t)$

for NL index $\rightarrow$ $-\frac{4\pi\omega_0^2}{c^2}\cdot 3\chi^{(3)}|A|^2 A$

$$\bar{n}_2 = \frac{3\pi\chi^{(3)}}{n_0} \rightarrow -\frac{4\omega_0^2 n_0}{c^2}\bar{n}_2|A|^2 A$$

$$\downarrow$$

$$2\bar{n}_2|A|^2 = n_2 I \qquad -\frac{2\omega_0^2 n_0}{c^2} n_2 I\, A$$

divide by $2ik_0 = \frac{2i\omega_0 n_0}{c}$

$$\rightarrow \text{RHS} = i\frac{\omega_0}{c} n_2 I A \equiv i\gamma|A|^2$$

put into wave eqn:

$$2ik_0 \frac{\partial}{\partial z} A - k_0 k_2 \frac{\partial^2}{\partial \tau^2} A = -\frac{4\omega_0^2 n_0 \bar{n}_2}{c^2}|A|^2 A$$

or $$\frac{\partial A}{\partial z} + \frac{ik_2}{2}\frac{\partial^2}{\partial \tau^2} A = i\gamma|A|^2 A \qquad \text{NLS}$$

$$\gamma = \frac{\omega_0 n_2}{c}$$

Nonlinear pulse propagation

Dynamics result from a balance of dispersion and nonlinearity.

define some characteristic lengths:

dispersion only

$$\frac{\partial A}{\partial z} + i\frac{k_2}{2}\frac{\partial^2 A}{\partial \tau^2} = 0$$

for a gaussian pulse $A \sim e^{-\tau^2/T^2}$

$$\partial_\tau A \simeq \frac{-2\tau}{T^2} A \qquad \partial_\tau^2 A = \frac{-2}{T^2} A - \frac{2\tau}{T^2}\left(-\frac{2\tau}{T^2}\right) A$$

$$= -\frac{2}{T^2}\left(1 - \frac{2\tau^2}{T^2}\right) A$$

evaluate at $\tau = T$

$\rightarrow$ scaling $\frac{1}{L_D} \sim \frac{k_2}{T^2}$ or $L_D = T^2/|k_2|$

this is the dispersion length.

Practical note: if pulse duration is described in terms of FWHM, $\rightarrow$ extra factor $T^2 = \frac{T_{FWHM}^2}{4\ln 2}$

nonlinearity only

$$\frac{\partial A}{\partial z} \sim i\gamma|A|^2 A = i k_0 n_2 I \cdot A$$

$$L_{NL} = 1/\gamma|A|^2 = \frac{1}{k_0 n_2 I}$$

if $L_D \ll L_{NL}$, pulse spreads out quickly, $I\downarrow$ and little SPM

$L_D \approx L_{NL}$ both effects work simultaneously

$L_D = \infty$ SPM only

L_D [illegible] chirping during SPM - and compression.

Self-phase modulation (SPM)

$n(t) = n_0 + n_2 I(t)$

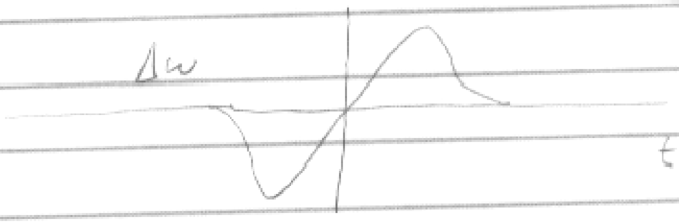
$E(z,t) = A(t) e^{i[k_0 n_0 + k_0 n_2 I(t)) z - \omega_0 t]}$ assuming no dispersion.

temporal phase is now

$$\phi(t) = k_0 n_2 I - \omega_0 t$$

local (instantaneous) frequency is

$$\omega_{inst} = -\frac{\partial \phi}{\partial t} = \omega_0 - k_0 n_2 \frac{\partial I}{\partial t}$$



$$\frac{\partial^2}{\partial z^2} E - \frac{n_0^2}{c^2} \frac{\partial^2}{\partial t^2} E = \frac{4\pi \cdot 3\chi^{(3)}}{c^2} \frac{\partial^2}{\partial t^2} \left(|E|^2 E \right)$$

assuming no dispersion. $\varepsilon(\omega) \to \varepsilon(\omega_0) = n_0^2$

with $E(z,t) = A(z,t) e^{i(kz - \omega_0 t)}$

$$\to \frac{\partial^2 A}{\partial z^2} + 2ik \frac{\partial A}{\partial z} - k^2 A - \frac{n_0^2}{c^2} \left(\frac{\partial^2 A}{\partial t^2} - 2i\omega_0 \frac{\partial A}{\partial t} - \omega_0^2 A \right)$$

$$= \frac{12\pi \chi^{(3)}}{c^2}$$