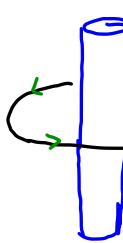


Problem 8.9

$b \gg a$ n turns/unit length



(a) Find induced current (I_r) in terms of $\frac{dI_s}{dt}$

Integral form of Faraday's Law

$$\oint \vec{E} \cdot d\vec{r} = - \frac{d\Phi_B}{dt} \quad \text{flux of the Solenoid's field}$$

EMF = ϵ

$$\epsilon = - \frac{d}{dt} \int_{\text{area of loop}} \vec{B}_{\text{solenoid}} \cdot d\vec{a} \quad B_{\text{solenoid}} = \mu_0 n I_s \quad S \gg a \\ = 0$$

$$= - \frac{d}{dt} (\mu_0 n I_s \pi a^2)$$

$$\epsilon = - \mu_0 n \pi a^2 \frac{dI_s}{dt} = I_r R$$

causes current to flow in ring

$$I_r = - \frac{\mu_0 n \pi a^2}{R} \frac{dI_s}{dt}$$

(b) The power ($I_r^2 R$) delivered to the ring must have come from the solenoid. Confirm this by calculating the Poynting vector just outside the solenoid (the electric field is due to the changing flux in the solenoid the magnetic field is due to the current in the ring). Integrate over the entire surface of the solenoid, and check that you recover the correct total power.

$$\int_{-\infty}^{\infty} \frac{b^2}{(b^2 + z^2)^{3/2}} dz = 2$$

\vec{B} ring : Biot-Savart Law

$$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{r} \times \hat{z}}{r^2}$$

$$\vec{B} = \frac{\mu_0 I r}{2} \frac{b^2}{(b^2 + z^2)^{3/2}} \hat{z}$$

from ring

Remember

Differential Form of
Pointing's Thm

$$\frac{\partial}{\partial t} (U_{\text{mech}} + U_{\text{em}}) = -\vec{\nabla} \cdot \vec{S}$$

$\downarrow \quad \downarrow$

mechanical electromagnetic
energy energy

$$= \frac{B^2}{2\mu_0} + \frac{\epsilon_0}{2} E^2$$

Compare to $\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot \vec{J}$

$\frac{1}{\mu_0} \vec{E} \times \vec{B}$

Integrate over a volume V

$$\int_V \frac{\partial}{\partial t} (U_{\text{mech}} + U_{\text{em}}) dV = - \int \vec{\nabla} \cdot \vec{S} dV$$

use divergence thm

$$\frac{d}{dt} \int_V (U_{\text{mech}} + U_{\text{em}}) dV = - \oint \vec{S} \cdot d\vec{a}$$

$$\frac{d}{dt} U_{\text{mech}} + \frac{d}{dt} U_{\text{em}} = - \oint \vec{S} \cdot d\vec{a}$$

$\downarrow \quad \downarrow$

total mechanical total electromagnetic
energy energy

time rate of change of
Power flowing the
out of the surface
energy

time rate of change of
the energy in the volume

Compare to

$$\frac{d}{dt} Q = - \oint \vec{J} \cdot d\vec{z}$$

charge flowing out
of surface

Forces

$$\begin{aligned} d\vec{F} &= d\vec{q} (\vec{E} + \vec{v} \times \vec{B}) \quad d\vec{q} = \rho d\vec{z} \\ &= \rho d\vec{z} (\vec{E} + \vec{v} \times \vec{B}) \\ \frac{d\vec{F}}{d\vec{z}} &= \rho \vec{E} + \cancel{\rho \vec{v} \times \vec{B}} \\ &= \rho \vec{E} + \vec{j} \times \vec{B} \end{aligned}$$

$$\vec{f} = \frac{d\vec{F}}{d\vec{z}} = \text{force per unit volume} = \rho \vec{E} + \vec{j} \times \vec{B}$$

Eliminate ρ & \vec{j} in terms of the fields Using Maxwell's Equations
and a bunch of vector calculus

$$\vec{T} = \vec{v} \cdot \vec{\tau} - \epsilon_0 \mu_0 \frac{\partial \vec{s}}{\partial t}$$

$\vec{\tau}$ = Maxwell Stress Tensor

$$T_{ij} = \epsilon_0 (E_i E_j - \frac{1}{2} \delta_{ij} E^2) + \frac{1}{\mu_0} (B_i B_j - \frac{1}{2} \delta_{ij} B^2)$$

Dot product of tensor w/ a vector gives another vector

$$(\vec{a} \cdot \vec{\tau})_j = a_i T_{ij} \quad \text{repeated index} \Rightarrow \text{sum}$$

$$(\vec{a} \cdot \vec{\tau})_z = a_x T_{xz} + a_y T_{yz} + a_z T_{zz}$$

To get the total force acting on the charges in the volume

$$\vec{F} = \int_V \vec{f} d\vec{z} = \int_V \vec{v} \cdot \vec{\tau} d\vec{z} - \epsilon_0 \mu_0 \int_V \frac{\partial \vec{s}}{\partial t} d\vec{z}$$

divergence Thm swap order

$$\vec{F} = \int_S \vec{\tau} \cdot d\vec{a} - \frac{d}{dt} \int_V \epsilon_0 \mu_0 \vec{s} d\vec{z}$$

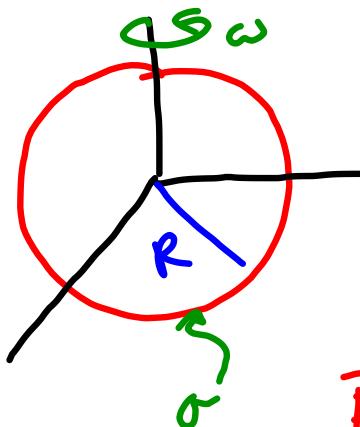
total force on
the charges in the volume

So if $\int \vec{s} d\vec{z}$ is time independent, like if we have static fields, total force is given only by the stress tensor over the surface.

T_{ij} is the force per area in the i direction acting on a surface element in the j direction

diagonal element (T_{xx}, T_{yy}, T_{zz})  pressure
off-diagonal element (T_{xy}, T_{xz} , etc.)  shear

Example (Problem 8.3)



Uniformly charged spherical shell

Find the force of magnetic attraction
between the N & S hemispheres

$$\vec{F} = \oint \vec{T} \cdot d\vec{a} - \frac{d}{dt} \int \mu_0 \epsilon_0 \vec{B} d\tau$$

$$\vec{B}_{\text{inside}} = \frac{2}{3} \mu_0 \sigma R \omega \hat{z}$$

$$\vec{B}_{\text{outside}} = \frac{\mu_0}{3} (\sigma R \omega) \left(\frac{R}{r}\right)^3 [2 \cos \theta \hat{r} + \sin \theta \hat{\theta}]$$

$$(\vec{T} \cdot d\vec{a})_z = T_{zx} da_x + T_{zy} da_y + T_{zz} da_z$$

Disk: magnetic part : $\frac{1}{\mu_0} [\vec{B} \cdot \vec{B}_j - \frac{1}{2} \delta_{ij} B^2]$