Separation of Variables

or: How I Learned to Stop Worrying and Love Boundary Value Problems

April 20, 2010

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Overview/Keywords/References

Advanced Engineering Mathematics **Slide Set Six**

Separation of Variables : Separation Constant, BVP, Half-Range Expansions

Reference Text: EK 12.3, 12.5

- See Also:
  - Lecture Notes : 13.LN.IntroToPDE
  - Lecture Notes : 14.LN.HeatEquation
  - Lecture Notes : 15.LN.WaveEquation

Before We Begin

Quote of Slide Set Six
Homer Simpson: From now on, there are three ways to do things: the right way, the wrong way, and the Max Power way.
Bart Simpson: Isn't that the wrong way?
Homer Simpson: Yes, but faster!
The Simpsons S10E13 : Homer to the Max (1999)

Problem Statement

Suppose we want to model the lateral flow of heat in a object L-units long, with initial temperature $f(x)$, whose endpoints are connected to a *heat bath* of constant temperature on a relative scale. The temperature evolution is well-modeled by,

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

$$(x,t) \in (0,L) \times (0,\infty), \quad c^2 = \frac{\kappa}{\rho\sigma}, \tag{2}$$

$$u(x,0) = f(x), \tag{3}$$

The interface conditions are modeled by,

$$u(0,t) = 0, \; u(L,t) = 0. \tag{4}$$

Question: Given (3) evolved by (1) subject to (4) find the temperature u at any point (x,t).

Solution Overview

The solution will be found by a three-step process:

1. Separation Step : Assume that the spatial component of u decouples from the temporal component. That is, we shall assume:

$$u(x,t) = F(x)G(t) \tag{5}$$

   Partial derivatives on u will be exchanged for ordinary derivatives on F and G. The PDE (1) will be traded for infinitely-many ODEs.

2. Solve the associated ODEs with boundary conditions (BVP).

3. Apply superposition (linearity) and Fourier methods to solve the initial value problem (IVP).

Step 1 : Separation of Variables - I

Assume/hope that $u(x,t) = F(x)G(t)$. Thus,

$$\frac{\partial u}{\partial t} = \partial_t(F(x)G(t)) = F(x)\partial_t G(t) = F(x)\frac{dG}{dt} = F(x)\dot{G}(t), \tag{6}$$

$$\frac{\partial^2 u}{\partial x^2} = \partial_{xx}(F(x)G(t)) = G(t)\frac{d^2 F}{dt^2} = G(t)F''(x), \tag{7}$$

which implies that,

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \iff \dot{G}(t)F(x) = c^2 G(t)F''(x), \tag{8}$$

and

$$\frac{\dot{G}(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)}, \tag{9}$$

for all $(x,t) \in (0,L) \times (0,\infty)$.

Step 1 : Separation of Variables - II

Notice that the LHS of (9) varies with respect to t while the RHS of (9) varies with respect to x. Here is the important argument:

- If these two sides are equal for all x and t then they must be equal to a function that has neither x's nor t's.

That is,

$$\frac{\dot{G}(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)} = -\lambda \in \mathbb{R}, \tag{10}$$

where λ is called the 'separation constant'. From this we have,

$$\dot{G} = -c^2 \lambda G, \tag{11}$$

$$F'' + \lambda F = 0. \tag{12}$$

Step 2 : Solving the ODE's

The temporal problem is easy:

$$\dot{G} = -c^2\lambda G \Rightarrow G(t) = \alpha e^{-c^2\lambda t}, \alpha \in \mathbb{R}. \quad (13)$$

The spatial problem is not as easy. Remember that the physical problem mandates the spatial interface conditions (4), which must be applied at this point. First, we note,

$$u(0,t) = F(0)G(t) = 0 \Rightarrow F(0) = 0 \text{ or } G(t) = 0, \quad (14)$$

$$u(L,t) = F(L)G(t) = 0 \Rightarrow F(L) = 0 \text{ or } G(t) = 0 \quad (15)$$

If $G(t) = 0$ for all t then $u(x,t) = F(x)G(t) = 0$ for all t. This is the trivial solution, which we ignore. Thus,

$$F(0) = 0, \;\; F(L) = 0. \quad (16)$$

Step 2 : Solving the BVP - I

We now have the boundary value problem,

$$F''(x) + \lambda F(x) = 0, \quad \lambda \in \mathbb{R} \tag{17}$$

$$F(0) = 0, \;\; F(L) = 0. \tag{18}$$

If $F(x) = e^{rx}$ then $F'' + \lambda F = e^{rx}(r^2 + \lambda) = 0$ and $r = \pm\sqrt{-\lambda}$.
We now have the three general solutions, which depend on λ:

$$\lambda > 0 : F_1(x) = c_1 e^{i\sqrt{\lambda}x} + c_2 e^{-i\sqrt{\lambda}x} = b_1 \sin(\sqrt{\lambda}x) + b_2 \cos(\sqrt{\lambda}x)$$

$$\lambda < 0 : F_2(x) = c_3 e^{\sqrt{|\lambda|}x} + c_4 e^{-\sqrt{|\lambda|}x} = b_3 \sinh(\sqrt{|\lambda|}x) + b_4 \cosh(\sqrt{|\lambda|}x)$$

$$\lambda = 0 : F_3 = c_5 e^0 + c_6 x e^0 = b_5 x + b_6$$

From these we must find all nontrivial functions, which also satisfy (18).

Step 2 : Solving the BVP - II

Geometry indicates $b_2 = b_3 = b_4 = b_5 = b_6 = 0$ and,

$$F(L) = b_1 \sin\left(\sqrt{\lambda} L\right) = 0 \Rightarrow b_1 = 0 \text{ or } \sin(\sqrt{\lambda} L) = 0. \tag{19}$$

Setting $b_1 = 0$ would imply that $F(x) = 0$ for all x and consequently $u(x,t)$ is trivial. Thus we require,

$$\sin(\sqrt{\lambda} L) = 0 \Rightarrow \sqrt{\lambda} L = n\pi,\ n = 1, 2, 3, \ldots, \tag{20}$$

which implies that there are countably-infinitely many λ's and functions that solve the BVP given by,

$$F_n(x) = b_n \sin\left(\sqrt{\lambda_n} x\right) = b_n \sin\left(\frac{n\pi}{L} x\right), \ \ n = 1, 2, 3, \ldots. \tag{21}$$

Step 2 : BVP - Details

Geometric arguments are always fast but often hide the details. Using algebra instead of geometry gives,

$$F_1(0) = b_1 \sin(\sqrt{\lambda} \cdot 0) + b_2 \cos(\sqrt{\lambda} \cdot 0) = b_2 = 0 \qquad \Rightarrow b_2 = 0,$$

$$F_2(0) = b_3 \sinh(\sqrt{|\lambda|} \cdot 0) + b_4 \cosh(\sqrt{|\lambda|} \cdot 0) = b_4 = 0 \qquad \Rightarrow b_4 = 0,$$

$$F_3(0) = b_5 \cdot 0 + b_6 = b_6 = 0 \qquad \Rightarrow b_6 = 0,$$

$$F_2(L) = b_3 \sinh(\sqrt{|\lambda|} L) = b_3 \left(\frac{e^{\sqrt{|\lambda|} L} - e^{-\sqrt{|\lambda|} L}}{2} \right) = 0 \qquad \Rightarrow b_3 = 0,$$

$$F_3(L) = b_5 L = 0 \qquad \Rightarrow b_5 = 0.$$

Step 3 : General Solution as a Linear Combination

We now have that $\sqrt{\lambda_n} = n\pi/L$, which implies temporal solutions given by,

$$G_n(t) = \alpha_n e^{-c^2\lambda_n t} = \alpha_n e^{-\left(\frac{cn\pi}{L}\right)^2 t}, \quad n = 1, 2, 3, \ldots, \tag{22}$$

and many solutions u_n to (1) given by,

$$u_n(x,t) = F_n(x)G_n(t) = b_n\alpha_n \sin\left(\sqrt{\lambda_n}x\right) e^{-c^2\lambda_n t} \tag{23}$$

$$= B_n \sin\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{cn\pi}{L}\right)^2 t}, \quad n = 1, 2, 3, \ldots. \tag{24}$$

Since (1) is linear the linear combination of solutions is also a solution. Thus the general solution is given by,

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{cn\pi}{L}\right)^2 t} \tag{25}$$

Step 3 : Application of the Initial Condition

We still have unknown constants B_n but we haven't used,

$$u(x,0) = f(x). \tag{26}$$

From the general solution we have,

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{cn\pi}{L}\right)^2 \cdot 0} \tag{27}$$

$$= \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right), \tag{28}$$

which is a Fourier sine series. Thus, we have that B_n are Fourier coefficients given by,

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx. \tag{29}$$

Conclusions

We have found the evolution of (3) modeled by (1) subject to (4). From this we note:

- The initial condition $f(x)$ has a Fourier sine half-range expansion.
- The general solution evolves each mode in the Fourier expansion of f by a factor of $e^{-c^2\lambda_n t}$.
- The separation assumption of $u(x,t) = F(x)G(t)$ was not too restrictive since λ_n shares information between the spatial and temporal components.
- The PDE (1) defines an infinite-dimensional space with a Fourier basis. Thus, solutions to (1) can represented as linear combination in this basis.