

# 10

## pulse propagation

## time dependence in frequency conversion

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# Representing an optical pulse in t and $\omega$ spaces

- Two ways to represent the field of a pulse:

- Time domain

$$E(t) = A(\mathbf{r}, t) \exp\left[i(\mathbf{k} \cdot \mathbf{r} - i \omega_0 t + \phi(t))\right] + c.c.$$

- Frequency domain

$$E(\omega) = FT\{E(t)\} = A(r, \omega - \omega_0) e^{i(\mathbf{k} \cdot \mathbf{r} + \varphi(\omega - \omega_0))} + A^*(r, \omega + \omega_0) e^{-i(\mathbf{k} \cdot \mathbf{r} - \varphi(\omega - \omega_0))}$$

- Both positive and negative frequency components: usually neglect negative side in linear optics

$$E(\omega) \approx A(r, \omega - \omega_0) \exp\left[i(\mathbf{k} \cdot \mathbf{r} + \varphi(\omega - \omega_0))\right]$$

- Both t and  $\omega$  representations contain the same information, same total energy.

- **Phase functions not the same in both domains**

- **Temporal phase:**  $\phi(t)$

- **Spectral phase:**  $\varphi(\omega)$

# Taylor expansion of spectral phase

- To simplify the phase, consider the first two terms

$$\varphi(\omega) = \varphi_0 + \varphi_1 \frac{\omega - \omega_0}{1!} + \varphi_2 \frac{(\omega - \omega_0)^2}{2!} + \dots$$

where  $\varphi_0 = \varphi(\omega_0)$  is the “**absolute phase**”

$$\varphi_1 = \left. \frac{d\varphi}{d\omega} \right|_{\omega=\omega_0}$$
 is the **group delay**.  $\tau_g(\omega) = \left. \frac{d\varphi}{d\omega} \right|_{\omega=\omega_0}$

$$\varphi_2 = \left. \frac{d^2\varphi}{d\omega^2} \right|_{\omega=\omega_0}$$
 is called the “**group-delay dispersion**.”

- In real situations, we sometimes have to include higher order phase, 3<sup>rd</sup>, 4<sup>th</sup>...

# Taylor expansion of temporal phase

- Here we expand around  $t=0$ , i.e. the center of the pulse

$$\phi(t) = \phi_0 + \phi_1 \frac{t}{1!} + \phi_2 \frac{t^2}{2!} + \dots$$

where  $\phi_0 = \phi(0)$  is the “**carrier-envelope**” or “**absolute phase**”

$$\phi_1 = \left. \frac{d\phi}{dt} \right|_{t=0}$$

the **instantaneous frequency is**  $\omega_{inst}(t) = -\frac{d\phi}{dt}$

$$\phi_2 = \left. \frac{d^2\phi}{dt^2} \right|_{t=0}$$

is called the “**temporal chirp.**”

- In real situations, we sometimes have to include higher order phase, 3<sup>rd</sup>, 4<sup>th</sup>...

# Intensity and phase of a Gaussian

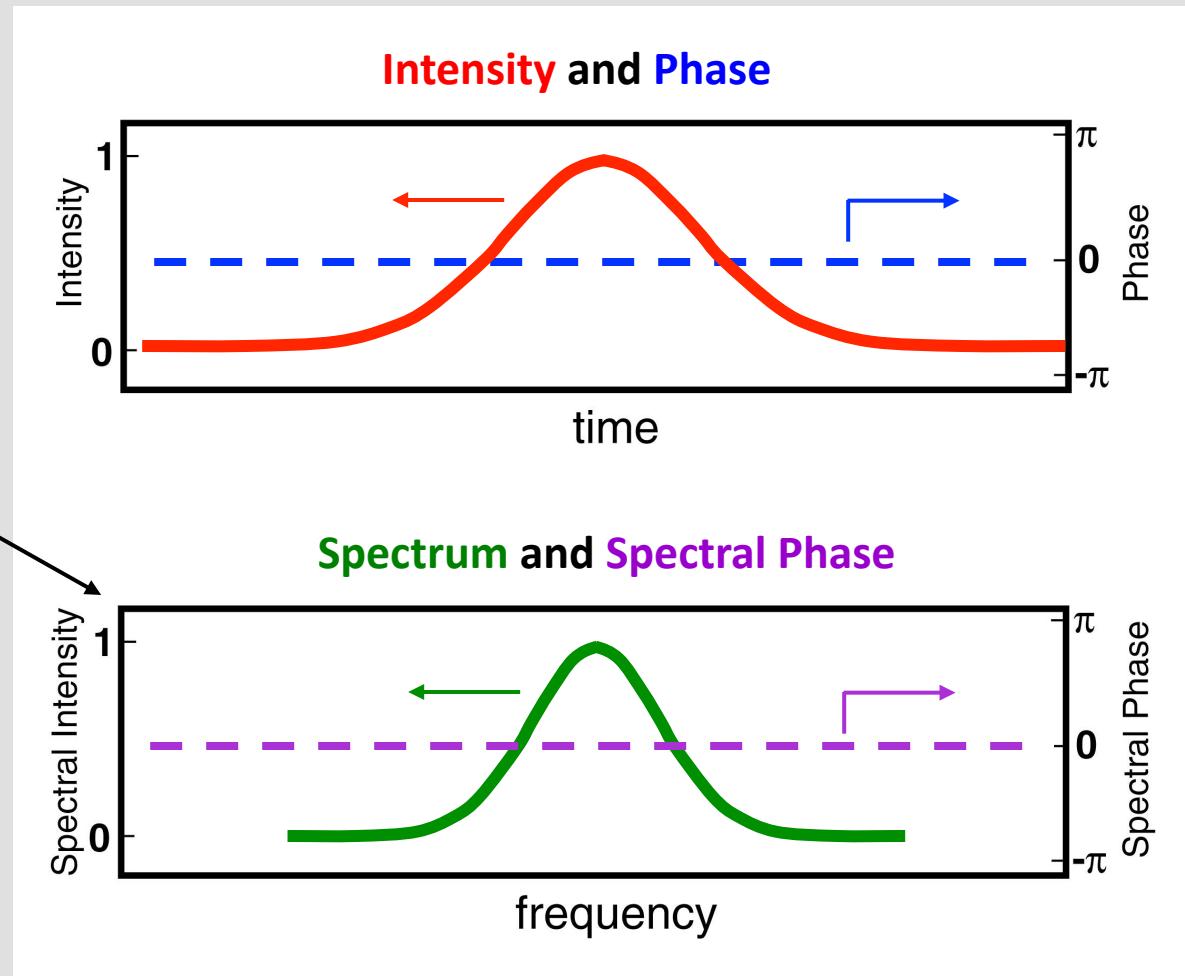
- The Gaussian is real, so its phase is zero in both domains.

Time domain:

A Gaussian  
transforms  
to a Gaussian

Frequency domain:

So the spectral phase is  
zero, too.

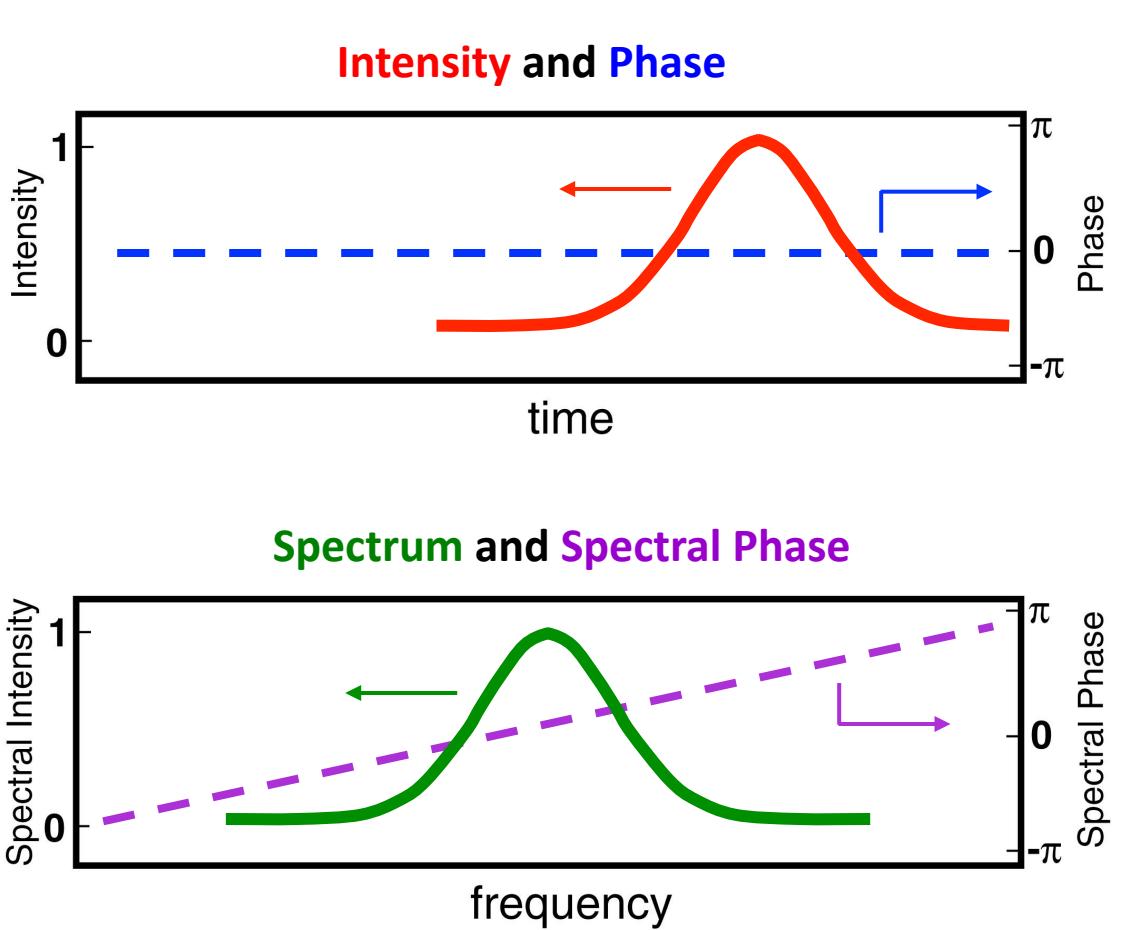


# The spectral phase of a time-shifted pulse

Recall the Shift Theorem:

$$FT\{f(t - t_0)\} = \exp(+i\omega t_0)F(\omega)$$

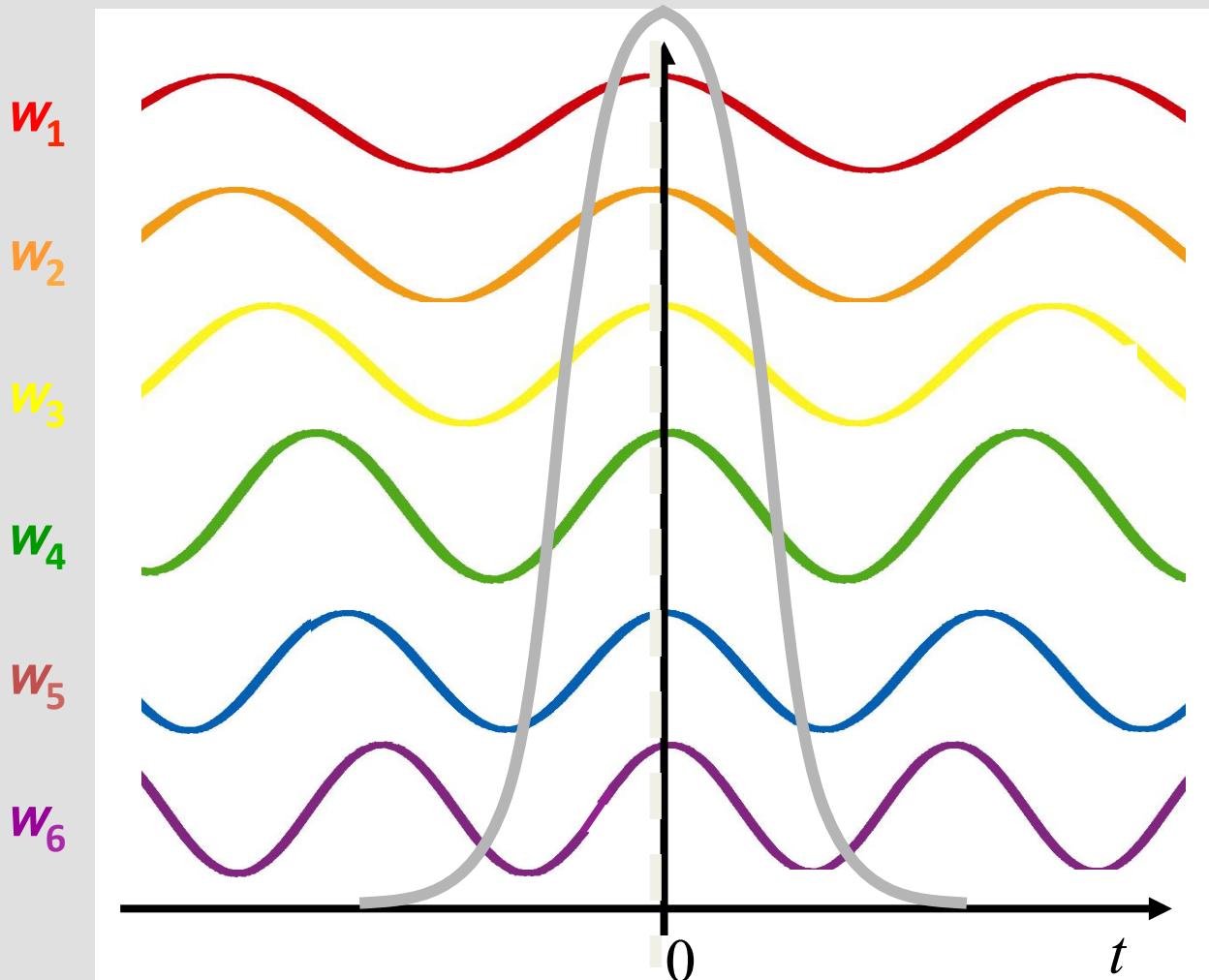
Time-shifted Gaussian pulse (with a flat phase):



So a time-shift simply adds some linear spectral phase to the pulse!

# What is the spectral phase?

The spectral phase is the phase of each frequency in the wave-form.



All of these frequencies have zero phase. So this pulse has:

$$\phi(\omega) = 0$$

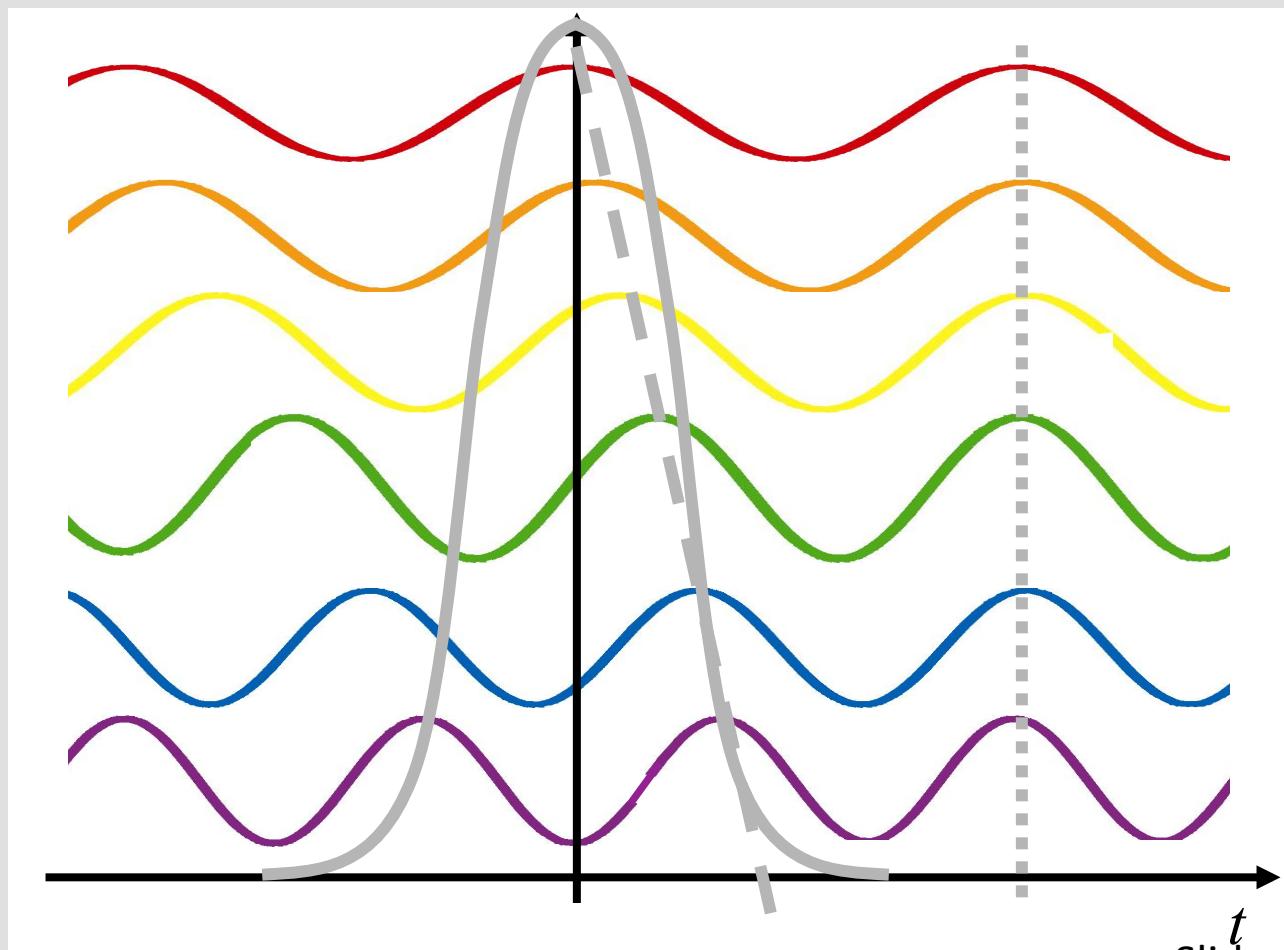
Note that this wave-form sees constructive interference, and hence peaks, at  $t = 0$ .

And it has cancellation everywhere else.

# Linear spectral phase: $\phi(\omega) = a\omega$ .

By the Shift Theorem, a linear spectral phase is just a delay in time.

And this is what occurs!



$$\varphi(\omega_1) = 0$$

$$\varphi(\omega_2) = 0.2 \pi$$

$$\varphi(\omega) = 0.4 \pi$$

$$\varphi(\omega_4) = 0.6 \pi$$

$$\varphi(\omega_5) = 0.8 \pi$$

$$\varphi(\omega_6) = \pi$$

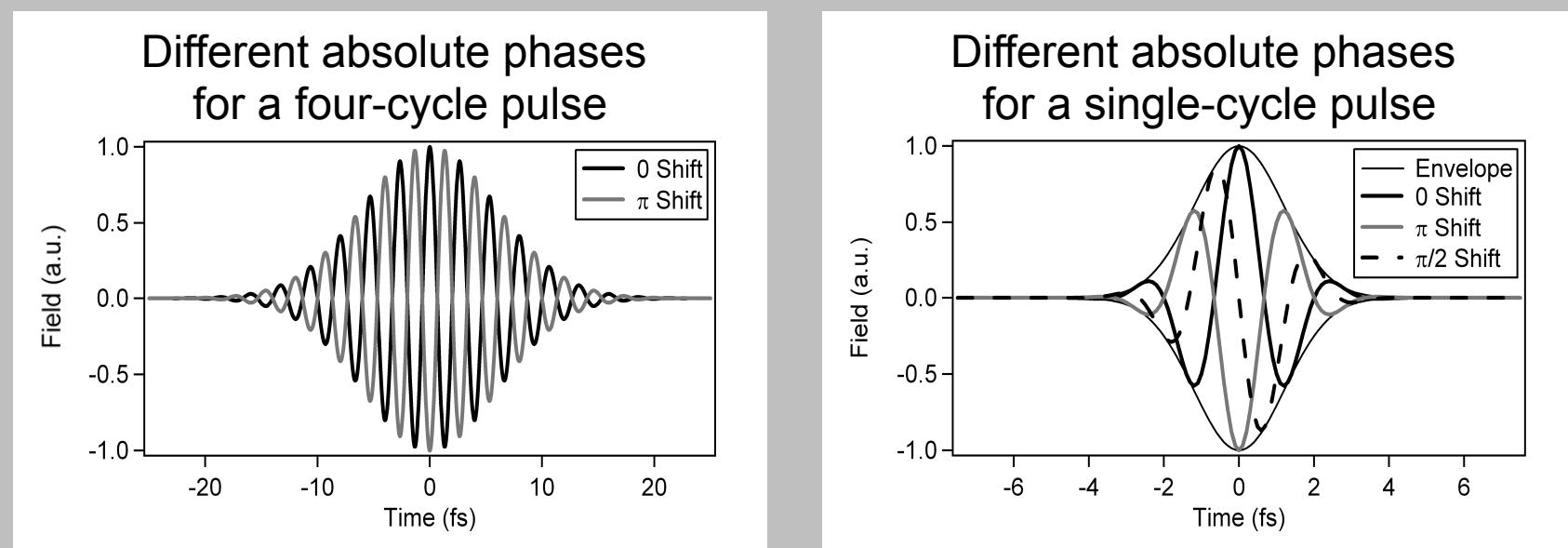
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# Zero<sup>th</sup>-order phase: the absolute phase

The absolute phase is the same in both the time and frequency domains.

$$f(t)\exp(i\phi_0) \rightarrow F(\omega)\exp(i\phi_0)$$

An absolute phase of  $\pi/2$  will turn a cosine carrier wave into a sine. It's usually irrelevant, unless the pulse is only a cycle or so long.



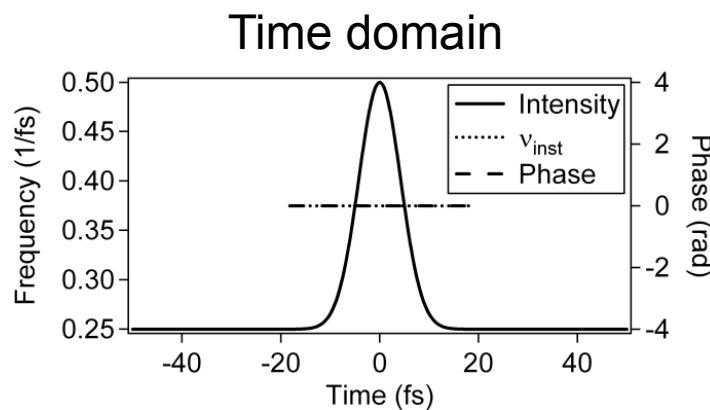
Notice that the two four-cycle pulses look alike, but the three single-cycle pulses are all quite different.

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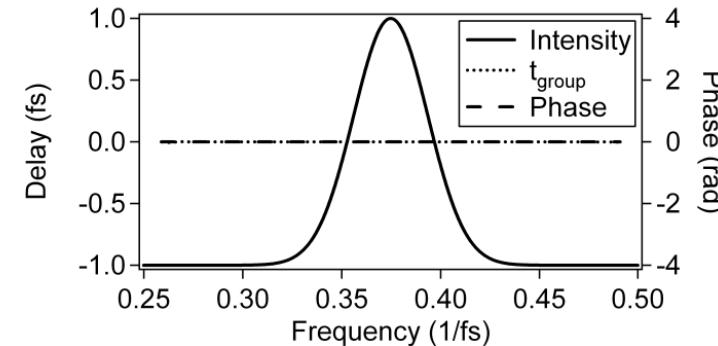
# First-order phase in frequency: a shift in time

By the Fourier-Transform Shift Theorem,  $F(\omega)\exp(i\omega\varphi_1) \rightarrow f(t - \varphi_1)$

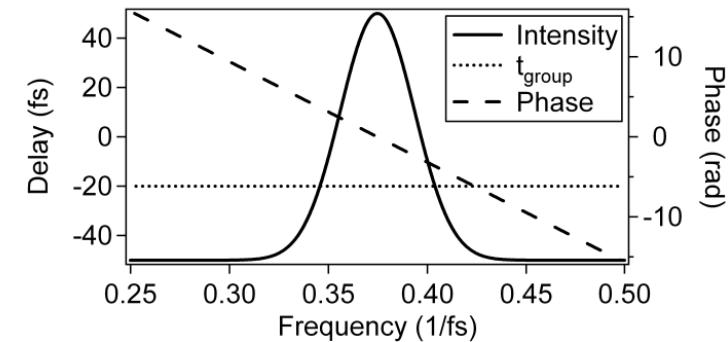
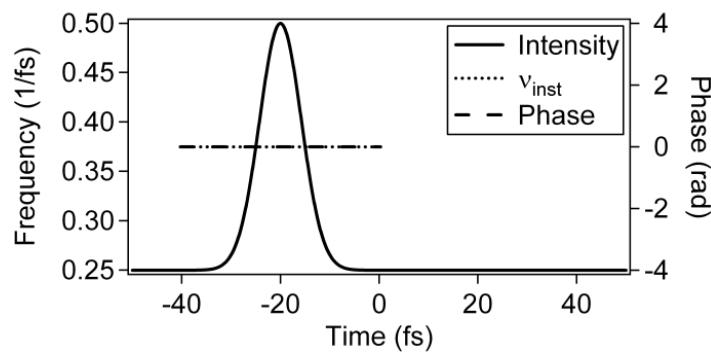
$$\varphi_1 = 0$$



Frequency domain



$$\varphi_1 = -20\text{ fs}$$



Note that  $\varphi_1$  does not affect the instantaneous frequency, but the group delay =  $\varphi_1$ .

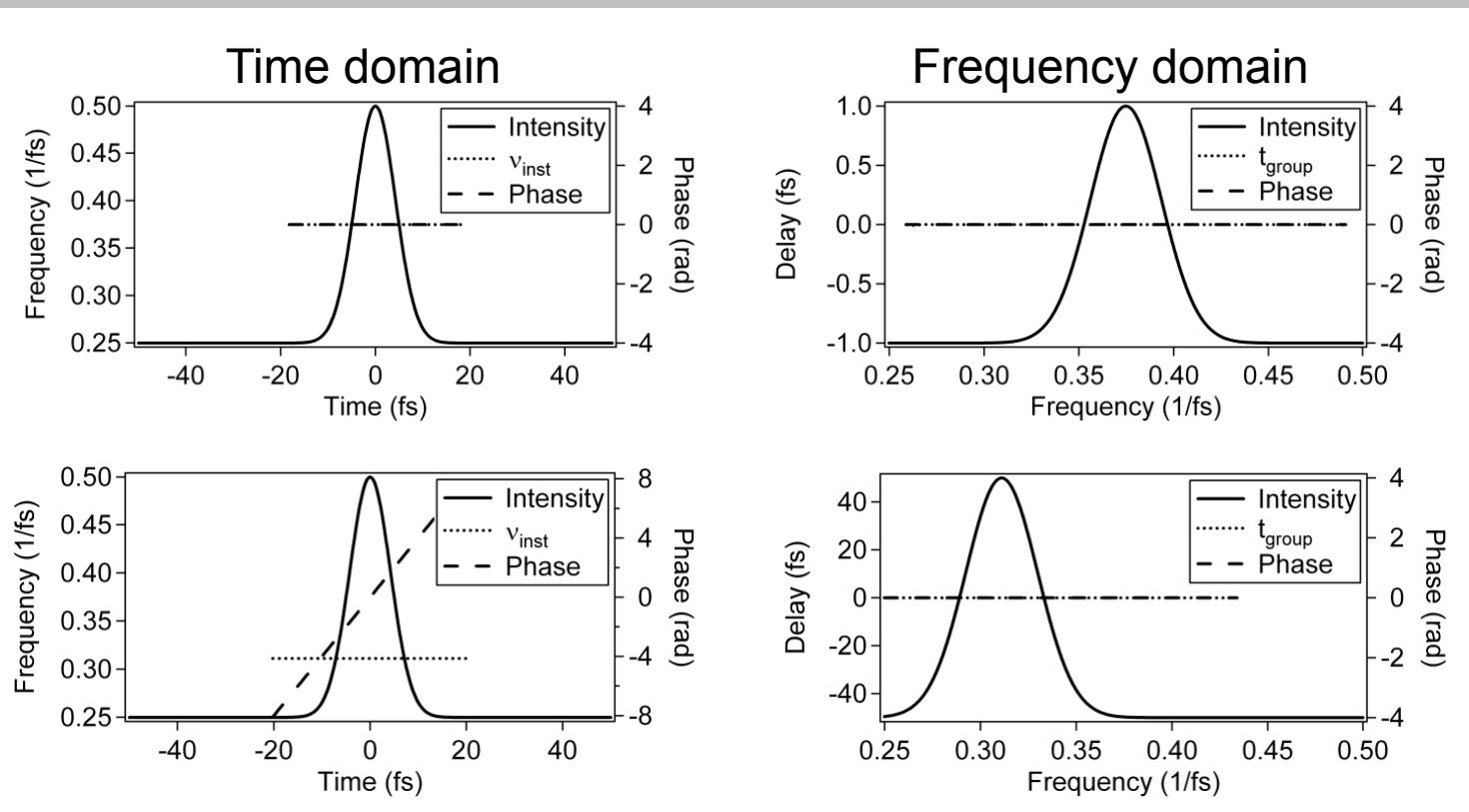
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# First-order phase in time: a frequency shift

By the Inverse-Fourier-Transform Shift Theorem,

$$f(t)\exp(-i\phi_1 t) \rightarrow F(\omega - \phi_1)$$

$$\phi_1 = 0/\text{fs}$$

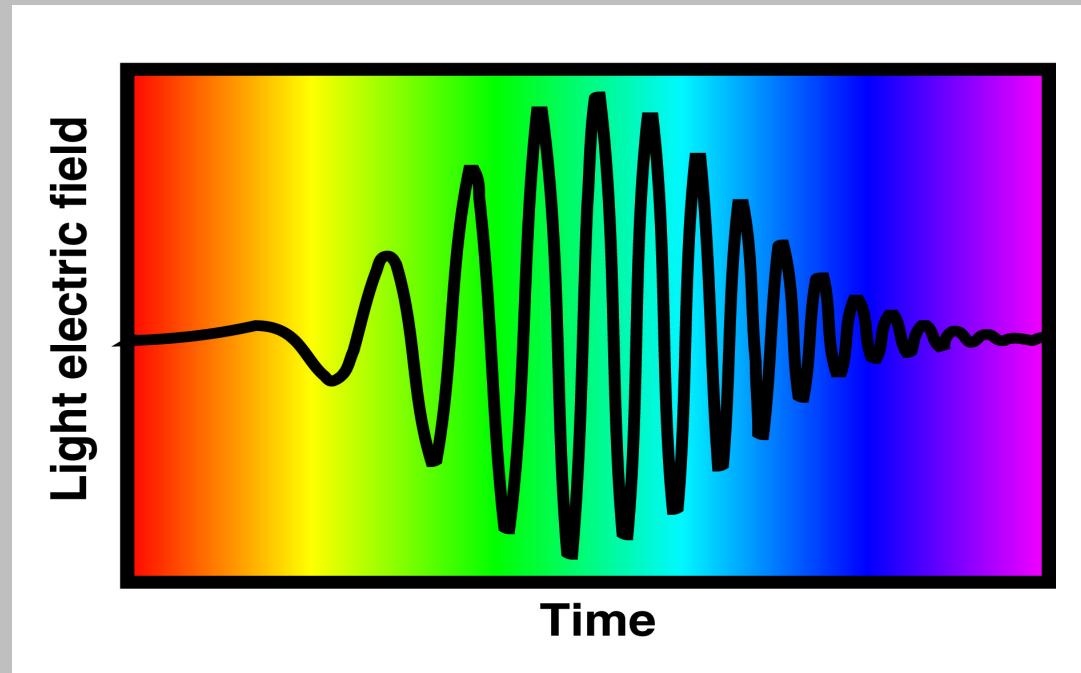


Note that  $\phi_1$  does not affect the group delay, but it does affect the instantaneous frequency.

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## Second-order phase: the linearly chirped pulse

A pulse can have a frequency that varies in time.

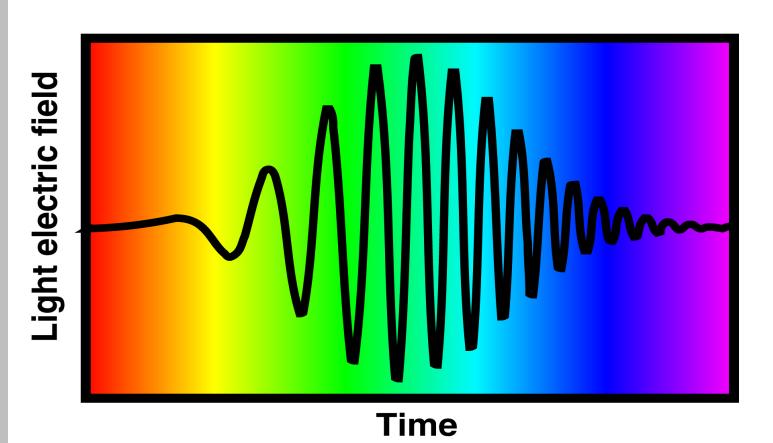


This pulse increases its frequency linearly in time (from red to blue).

In analogy to bird sounds, this pulse is called a "chirped" pulse.

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# The linearly chirped Gaussian pulse



We can write a linearly chirped Gaussian pulse mathematically as:

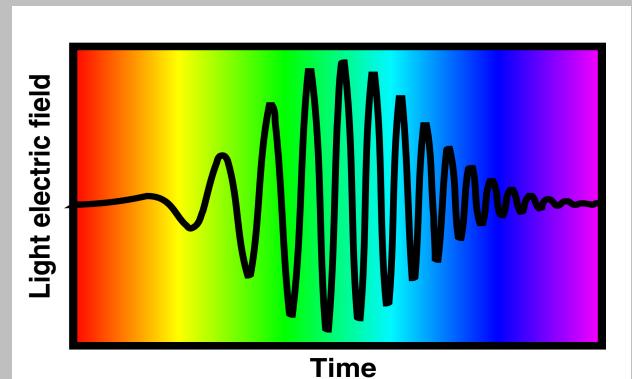
$$\begin{aligned} E(t) &= A(t) \exp[i\phi(t)] \\ &= E_0 \exp[-(t/\tau_G)^2] \exp[-i(\omega_0 t + \beta t^2)] \end{aligned}$$

—————  
↑ Gaussian amplitude      ↑ Carrier wave      ↑ Chirp

Note that for  $\beta > 0$ , when  $t < 0$ , the two terms partially cancel, so the phase changes slowly with time (so the frequency is low). And when  $t > 0$ , the terms add, and the phase changes more rapidly (so the frequency is larger).

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# The instantaneous frequency vs. time for a chirped pulse



A chirped pulse has:

$$E(t) \propto \exp\left[i(-\omega_0 t + \phi(t))\right]$$

where:

$$\phi(t) = -\beta t^2 \quad (\text{note the sign change})$$

The instantaneous frequency is:  $\omega_{inst}(t) \equiv \omega_0 - d\phi / dt$

which is:

$$\omega_{inst}(t) = \omega_0 + 2\beta t$$

So the frequency increases linearly with time. This is *positive chirp*.

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# The negatively chirped pulse

We have been considering a pulse whose frequency *increases* linearly with time: a *positively* chirped pulse.

One can also have a *negatively* chirped (Gaussian) pulse, whose instantaneous frequency *decreases* with time.

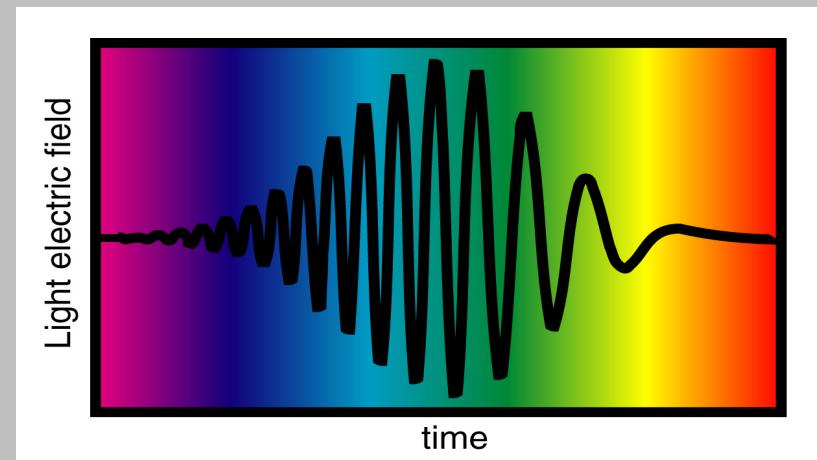
We simply allow  $\beta$  to be *negative* in the expression for the pulse:

$$\begin{aligned} E(t) &= E_0 \exp\left[-(t/\tau_G)^2\right] \exp\left[-i(\omega_0 t + \beta t^2)\right] \\ &= E_0 \exp\left[-(t/\tau_G)^2\right] \exp\left[-i(\omega_0 t - |\beta| t^2)\right] \end{aligned}$$

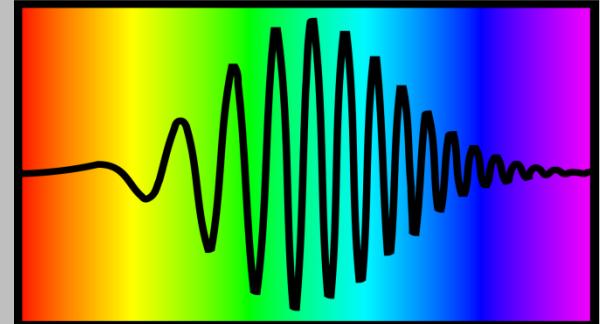
And the instantaneous frequency will decrease with time:

$$\omega_{inst}(t) = \omega_0 + 2\beta t = \omega_0 - 2|\beta|t$$

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# The Fourier transform of a chirped pulse



Writing a linearly chirped Gaussian pulse as:

$$\mathcal{E}(t) \propto E_0 \exp[-\alpha t^2] \exp[-i(\omega_0 t + \beta t^2)] + c.c \quad \text{where} \quad \alpha \propto 1/\Delta t^2$$

or:

$$\mathcal{E}(t) \propto E_0 \exp[-(\alpha + i\beta)t^2] \exp[-i\omega_0 t] + c.c.$$

**A Gaussian with a complex width!**

Fourier-Transforming yields:

$$\tilde{E}(\omega) \propto E_0 \exp\left[-\frac{1/4}{\alpha + i\beta} (\omega - \omega_0)^2\right]$$

neglecting the negative-frequency term due to the c.c.

**A chirped Gaussian pulse Fourier-Transforms to itself!!!**

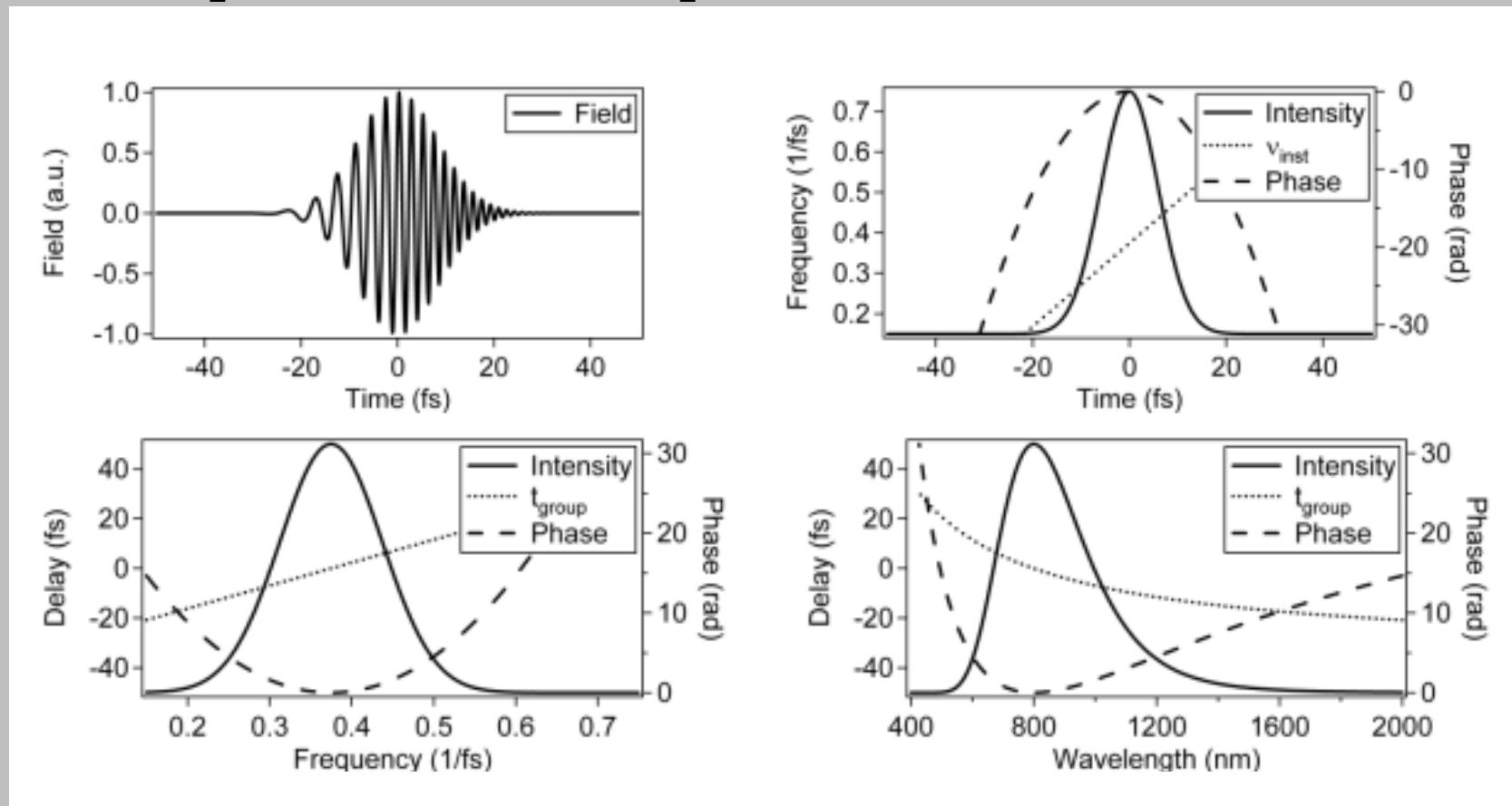
Rationalizing the denominator and separating the real and imag parts:

$$\tilde{E}(\omega) \propto E_0 \exp\left[-\frac{\alpha/4}{\alpha^2 + \beta^2} (\omega - \omega_0)^2\right] \exp\left[+i\frac{\beta/4}{\alpha^2 + \beta^2} (\omega - \omega_0)^2\right]$$

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## 2<sup>nd</sup>-order phase: positive linear chirp

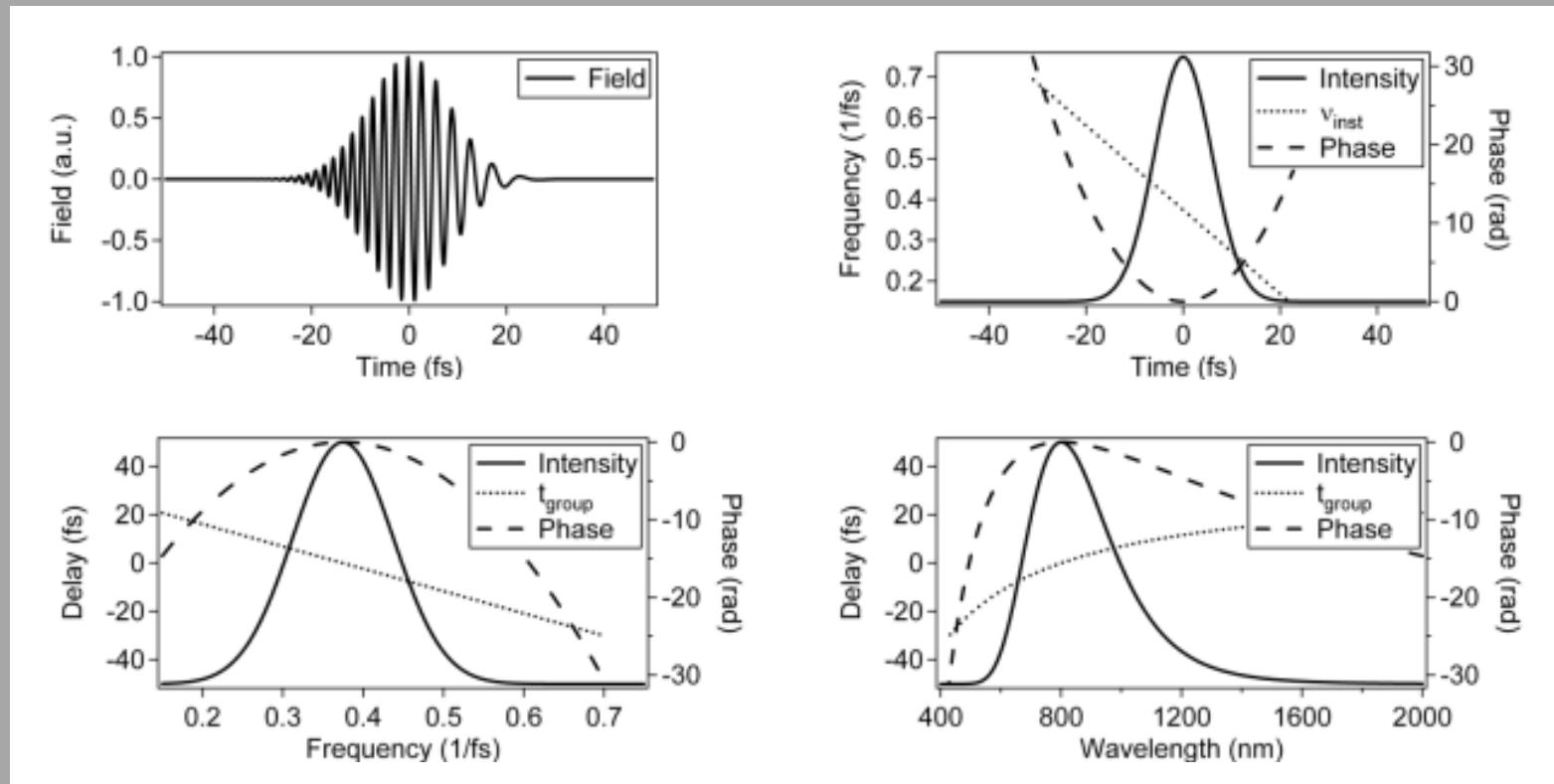
Numerical example: Gaussian-intensity pulse w/ positive linear chirp,  $\phi_2 = -0.032 \text{ rad/fs}^2$  or  $\varphi_2 = 290 \text{ rad fs}^2$ .



Here the quadratic phase has stretched what would have been a 3-fs pulse (given the spectrum) to a 13.9-fs one. Slide modified from R. Trebino

## 2<sup>nd</sup>-order phase: negative linear chirp

Numerical example: Gaussian-intensity pulse w/ negative linear chirp,  $\phi_2 = +0.032 \text{ rad/fs}^2$  or  $\varphi_2 = -290 \text{ rad fs}^2$ .



As with positive chirp, the quadratic phase has stretched what would have been a 3-fs pulse (given the spectrum) to a 13.9-fs one.

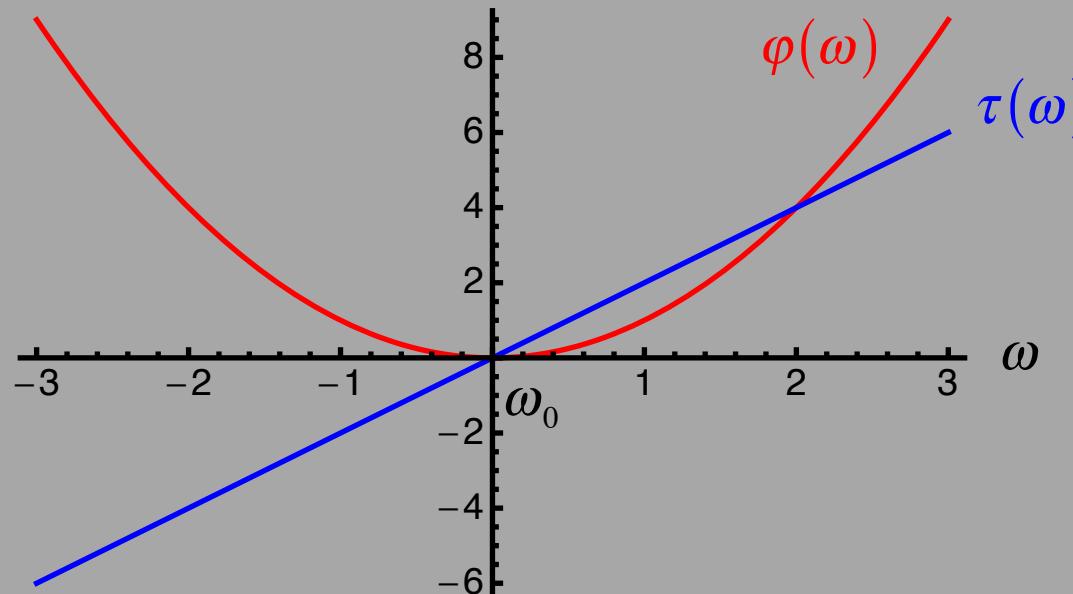
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# Group delay vs spectral phase

- The group delay gives the arrival time of the different frequency components
- So a positive 2<sup>nd</sup> order phase gives a positive slope to the group delay:

$$\tau_g(\omega) = \frac{d\varphi}{d\omega}$$

$$\varphi(\omega) = \varphi_0 + \varphi_1 \frac{\omega - \omega_0}{1!} + \varphi_2 \frac{(\omega - \omega_0)^2}{2!} + \dots$$

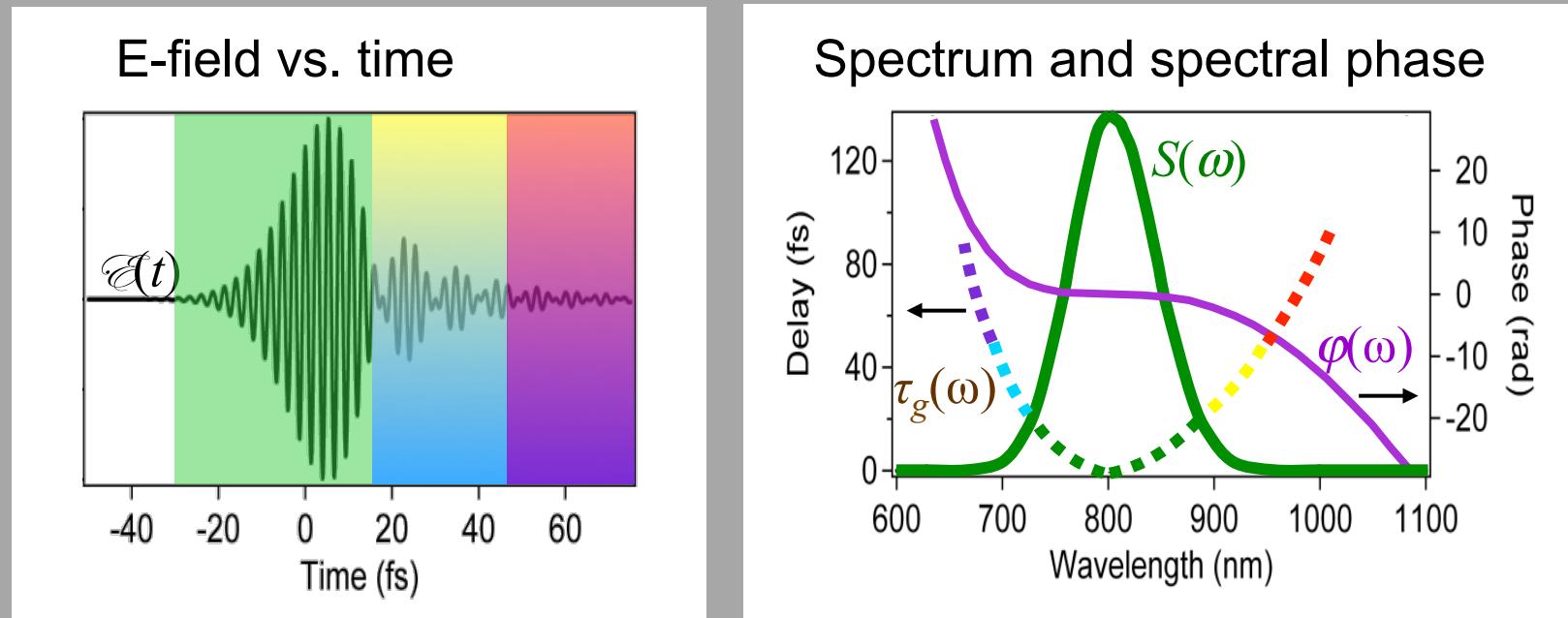


Not usually important:  
• phase constant  
• group delay shift

**Use group delay variation to visualize chirp.**

# 3<sup>rd</sup>-order spectral phase: quadratic chirp

The red and blue colors coincide in time and interfere.

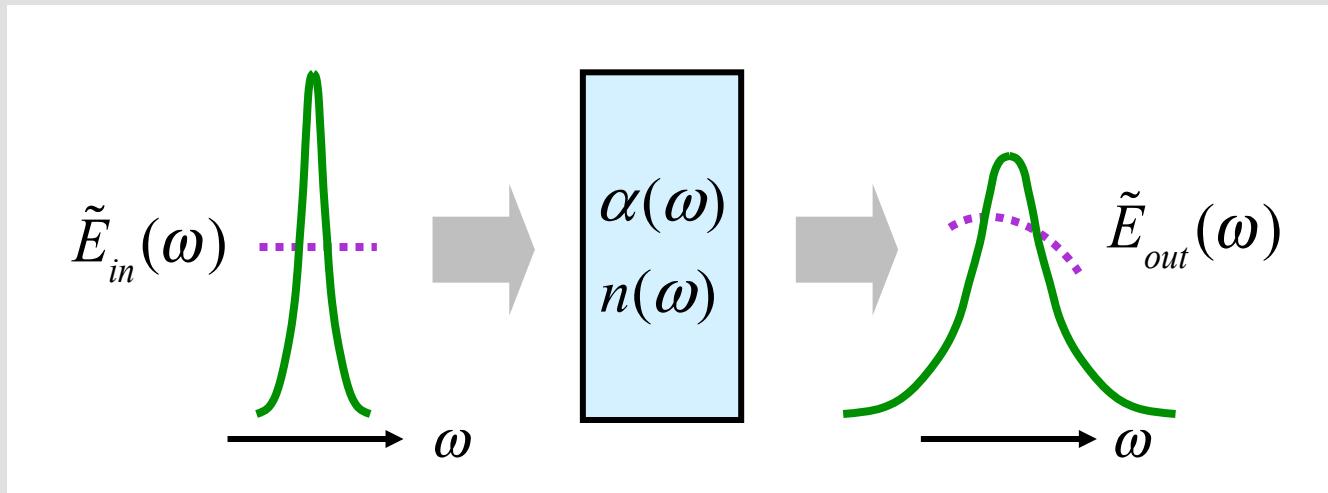


Trailing satellite pulses in time indicate positive spectral cubic phase, and leading ones indicate negative spectral cubic phase.

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# Pulse propagation

- What happens to a pulse as it propagates through a medium?
- Always model (linear) propagation in the **frequency domain**. Also, you must know the entire field (i.e., the intensity and phase) to do so.

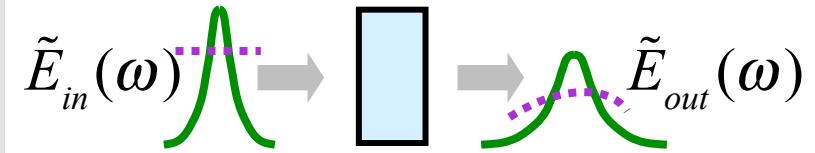


$$\tilde{E}_{out}(\omega) = \tilde{E}_{in}(\omega) \exp\left[-\frac{\alpha(\omega)}{2}L\right] \exp[i k(\omega) L]$$

In the time domain, propagation is a convolution—much harder.

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## Pulse propagation (continued)



Rewriting this expression using  $k = n(\omega) \omega/c$ :

$$\tilde{E}_{out}(\omega) = \tilde{E}_{in}(\omega) \exp[-\alpha(\omega)L/2] \exp[i\omega n(\omega)L/c]$$

Separating out the spectrum and spectral phase:

$$S_{out}(\omega) = S_{in}(\omega) \exp[-\alpha(\omega)L]$$

$$\varphi_{out}(\omega) = \varphi_{in}(\omega) + n(\omega) \frac{\omega}{c} L$$

Absorption (or gain) modifies the spectral amplitude,  
Refractive index modifies the spectral phase

# Propagation of a Gaussian pulse

- Start with pulse in t-domain

$$E(z=0,t) = A_0 e^{-t^2/t_0^2} e^{-i\omega_0 t}$$

- FT to frequency space:

$$E(z=0,\omega) = \text{FT}\{E(t)\} = A_0 t_0 e^{-\frac{1}{4}(\omega-\omega_0)^2 t_0^2}$$

- Apply phase shift.

$$\begin{aligned} E(z,\omega) &= A_0 t_0 e^{-\frac{1}{4}(\omega-\omega_0)^2 t_0^2} e^{i\frac{\omega}{c}n(\omega)z} \approx A_0 t_0 e^{-\frac{1}{4}(\omega-\omega_0)^2 t_0^2} e^{i\left(\varphi_0 + (\omega - \omega_0)\varphi_1 + \frac{1}{2}(\omega - \omega_0)^2 \varphi_2\right)} \\ &= A_0 t_0 e^{i\varphi_0} \exp\left[i(\omega - \omega_0)\varphi_1\right] \exp\left[-(\omega - \omega_0)^2 \left(\frac{t_0^2}{4} - i\frac{1}{2}\varphi_2\right)\right] \end{aligned}$$

Constant phase      Group delay shift      Chirp?

- Now inverse transform to t-domain.

# Propagated pulse in time domain

- In the time-domain, pulse can be written

$$E(z,t) = A_0 t_0 \frac{1}{2\pi} \int e^{i\varphi_0} \exp\left[i(\omega - \omega_0)\varphi_1\right] \exp\left[-(\omega - \omega_0)^2 \left(\frac{t_0^2}{4} - i\frac{1}{2}\varphi_2\right)\right] e^{-i\omega t} d\omega$$

- We will use the shift theorem for carrier and group delay, so consider this integral:

$$f(t) = \frac{1}{2\pi} \int \exp\left[-\delta\omega^2 \left(\frac{t_0^2}{4} - i\frac{1}{2}\varphi_2\right)\right] e^{-i\delta\omega t} d\delta\omega$$

- So that

$$E(z,t) = A_0 t_0 e^{i\varphi_0 - i\omega_0 t} f(t - \varphi_1)$$

- Note that the group delay is just the transit time through

$$\varphi_1 = \tau_g(\omega_0) = \frac{d\varphi}{d\omega} \Big|_{\omega=\omega_0} = \frac{dk}{d\omega} \Big|_{\omega=\omega_0} \cdot L = \frac{L}{v_g}$$

# Chirped output pulse

- We're doing the FT of a complex Gaussian

$$f(t) = \frac{1}{2\pi} \int \exp\left[-\delta\omega^2 \left(\frac{t_0^2}{4} - i\frac{1}{2}\varphi_2\right)\right] e^{-i\delta\omega t} d\delta\omega$$

$$FT^{-1}\left\{\exp\left(-T^2\omega^2/4\right)\right\} = \frac{1}{\sqrt{\pi T^2}} \exp\left(-t^2/T^2\right) \quad T^2 = t_0^2 - 2i\varphi_2$$

$$f(t) = \frac{1}{\sqrt{\pi(t_0^2 - 2i\varphi_2)}} \exp\left(-\frac{t^2}{t_0^2 - 2i\varphi_2}\right)$$

$$\frac{1}{t_0^2 - 2i\varphi_2} = \frac{t_0^2 + 2i\varphi_2}{t_0^4 + 4\varphi_2^2} = \frac{1 + \frac{2i\varphi_2}{t_0^2}}{t_0^2 \left(1 + \left(\frac{2\varphi_2}{t_0^2}\right)^2\right)}$$

# Chirped output pulse

- The pulse duration and chirp parameter vary with z

z-dependent pulse duration

$$\tau(z) = t_0 \sqrt{1 + \left( \frac{2\varphi_2}{t_0^2} \right)^2} = t_0 \sqrt{1 + \left( \frac{2k_2}{t_0^2} z \right)^2}$$

z-dependent chirp parameter

$$\varphi_2(z) = \frac{d^2\varphi}{d\omega^2} \Big|_{\omega=\omega_0} = z \frac{d^2k}{d\omega^2} \Big|_{\omega=\omega_0} = k_2 z$$

$$\beta(z) = \frac{1}{\tau^2(z)} \frac{2\varphi_2}{t_0^2}$$

$$f(t) = \frac{1}{\sqrt{\pi} \tau(z)} \exp\left(-\frac{t^2}{\tau^2(z)}\right) \exp(-i\beta t^2)$$

- This dispersion dependence is just like a Gaussian beam that focuses and diverges.

# Phase mismatch bandwidth and group walkoff

- Phase matching condition for SHG:  $\Delta k = 2k_1 - k_2$
- Even if there is zero phase mismatch for the carrier, it won't be matched for the whole bandwidth:

$$\Delta k(\omega_1) = 2k_1(\omega_1) - k_2(2\omega_1)$$

$$\Delta k(\omega_1) \approx \Delta k + (\omega - \omega_{10}) \left[ 2\partial_{\omega_1} k_1 \Big|_{\omega=\omega_{10}} - 2\partial_{\omega_1} k_2 \Big|_{\omega=2\omega_{10}} \right]$$

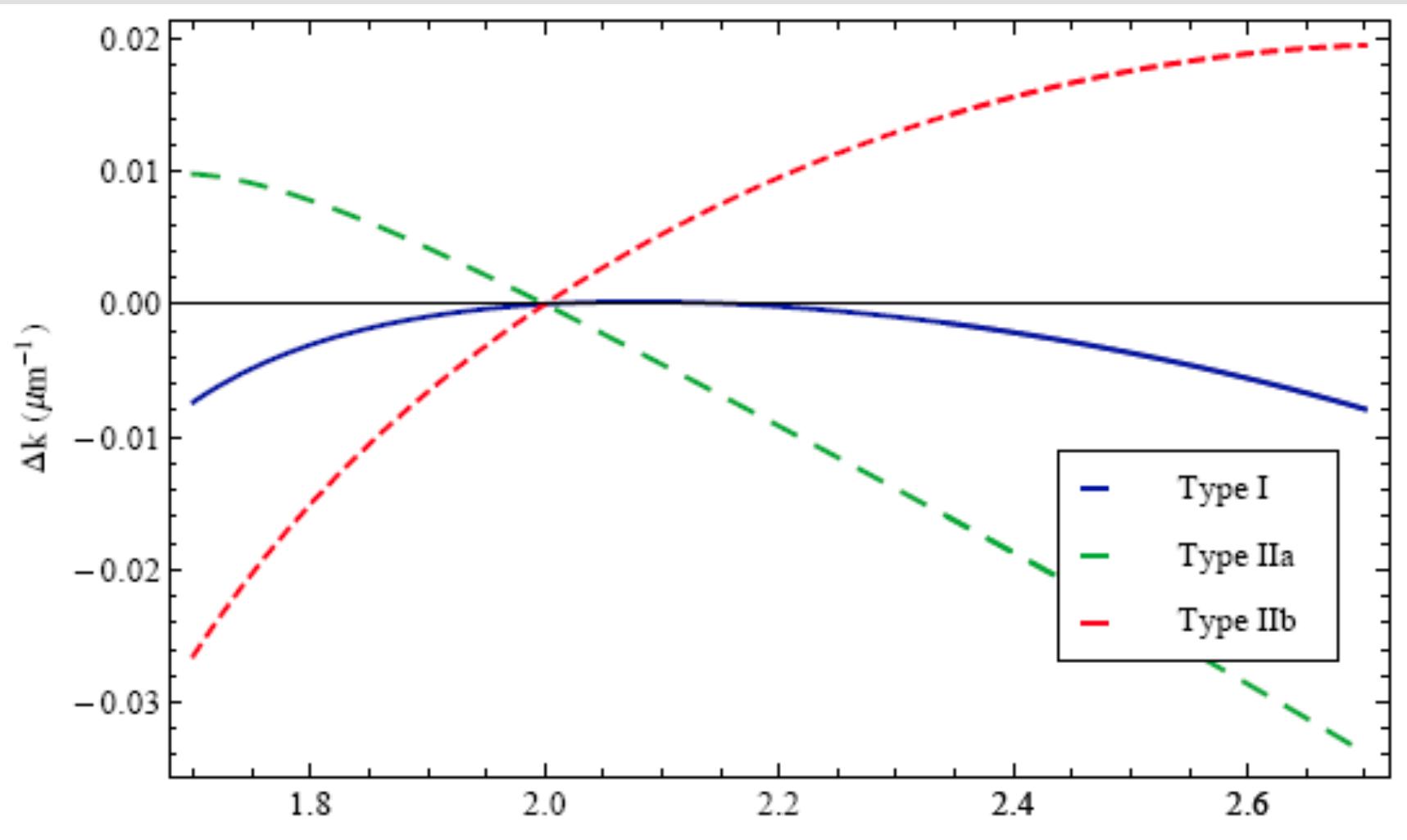
$$\Delta k(\omega_1)L \approx \Delta kL + 2(\omega - \omega_{10})(\varphi_1(\omega_{10}) - \varphi_1(\omega_{20}))$$

Phase mismatch  
normally = 0

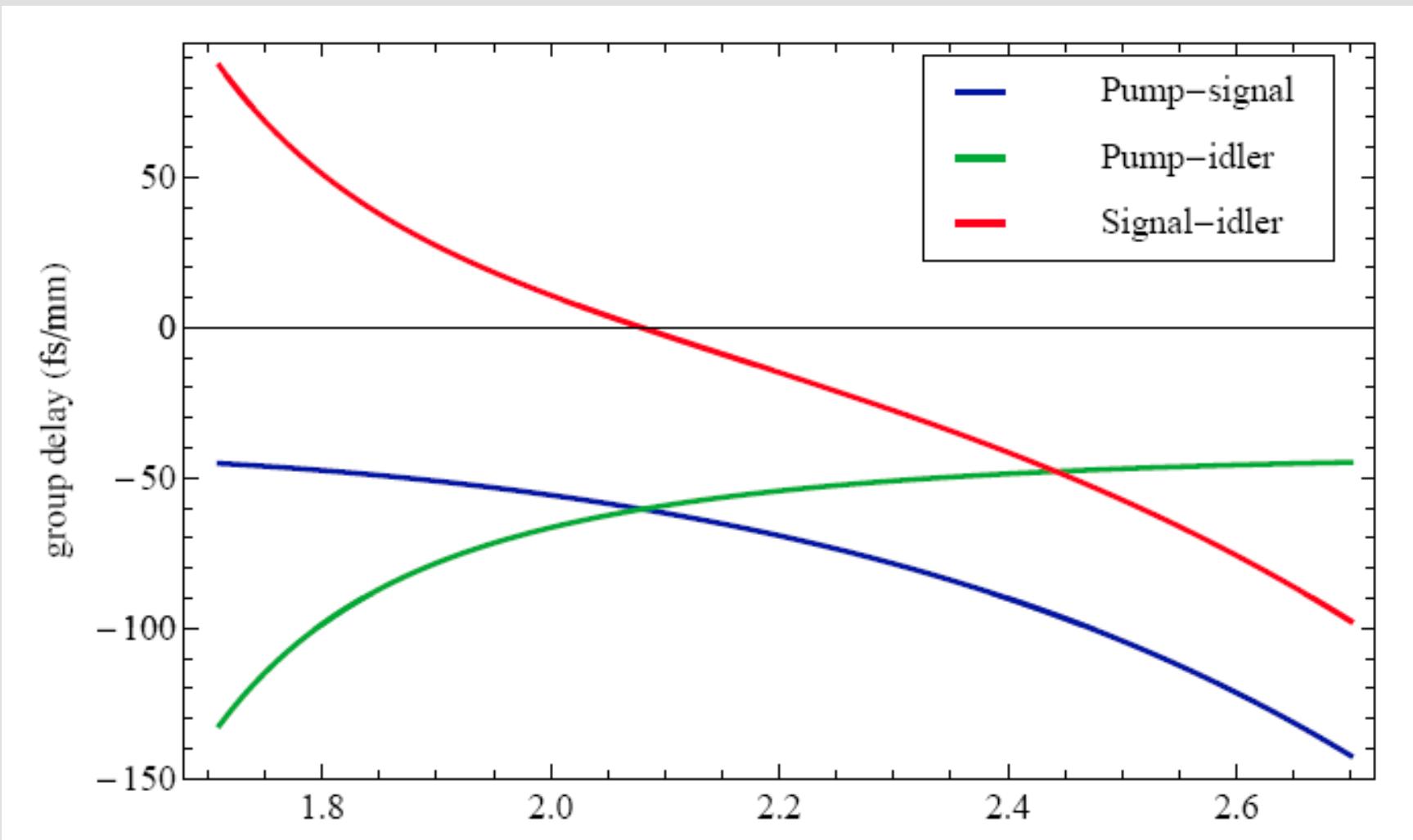
Group delay walkoff

- So the phase matching bandwidth is directly connected to the group walkoff

# Phase matching bandwidth in OPA



# Group velocity mismatch in OPA



# Maximum BW at zero dispersion point

Zero second-order dispersion wavelengths (nm)

BBO ordinary	1487
LBO y-axis	1203
LBO z-axis	1208
BiBO y-axis	1598
KTP x-axis	1683
KTP y-axis	1688
KTP z-axis	1789
KTA x-axis	1941
KTA y-axis	1912
KTA z-axis	1942

Signal/idler GVD at degeneracy vs. pump wavelength

