

Key

In order to receive full credit, SHOW ALL YOUR WORK. Full credit will be given only if all reasoning and work is provided. When applicable, please enclose your final answers in boxes.

1. (10 Points) True/False and Short Response

(a) Mark each statement as either true or false. No justification is needed.

i. Suppose  $f$  is a periodic function such that  $f(-x) = -f(x)$ . The Fourier series representation of  $f$  will have only cosine functions. False Odd  $\Rightarrow$  Sine Series

ii. Every real Fourier series can be written as a complex Fourier series.

True See 10.4

iii. The periodic extension,  $f^*(x)$ , of  $f(x)$  is unique.

False see 10.3

iv. The Fourier transform of a function with even symmetry has an odd symmetry.

False, it would have even symmetry.

v. If a periodic function is neither odd nor even then the Fourier series representation will have both sine and cosine functions.

True

(b) Explain why every piecewise continuous function defined on a bounded domain of  $\mathbb{R}$  has a Fourier series representation.

The  $f_{xy}$  can be repeated throughout space by periodic extension. This new  $f_{xy}$  which is equal to the original  $f_{xy}$  on its domain would then be periodic and thus have a F.S. Rep.

(c) How is the Fourier integral related to the Fourier series? What is the purpose of each?

a F.I. is like a F.S under the limit  $L \rightarrow \infty$ .

If a F.S. is used to Rep. a periodic  $f_{xy}$

then a F.I. would be used to Rep.

a non-periodic  $f_{xy}$

2. (10 Points) Given the following integrals,

$$2\pi \cdot a_0 = 5 = \int_{-\pi}^{\pi} f(x) dx, \quad \frac{4}{n\pi} (1 - \cos(2n\pi)) = \int_{-\pi}^{\pi} f(x) \cos(nx) dx, \quad \frac{2}{n^2} \sin(n\pi) = \int_{-\pi}^{\pi} f(x) \sin(nx) dx,$$

$$2\pi \cdot c_n = i \frac{(-1)^n}{n} = \int_{-\pi}^{\pi} g(x) e^{-inx} dx, \quad b_n = \frac{e^{in\pi} - e^{-in\pi}}{4\pi} = \int_{-\pi}^{\pi} g(x) dx, \quad \frac{e^{i\omega} - e^{-i\omega}}{2i\omega} = \int_{-\infty}^{\infty} h(x) e^{-i\omega x} dx.$$

(a) Calculate the real Fourier series of  $f(x)$ .

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) = a_0 = \frac{5}{2\pi}$$

(b) Calculate the complex Fourier series of  $g(x)$ .

$$g(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{i(-1)^n}{2\pi n} e^{inx}$$

(c) Calculate the Fourier transform of  $h(x)$ .

$$\hat{h}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(x) e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} \frac{e^{i\omega} - e^{-i\omega}}{2i\omega} = \frac{1}{\sqrt{2\pi}} \frac{\sin(\omega)}{2i\omega} = \frac{\sin(\omega)}{\sqrt{2\pi} \omega}$$

(d) Determine the symmetry of the function  $f(x)$ .

Even

(e) Determine the symmetry of the function  $g(x)$ .

$c_n \equiv \text{ing} \Rightarrow \text{odd}$  (See g)

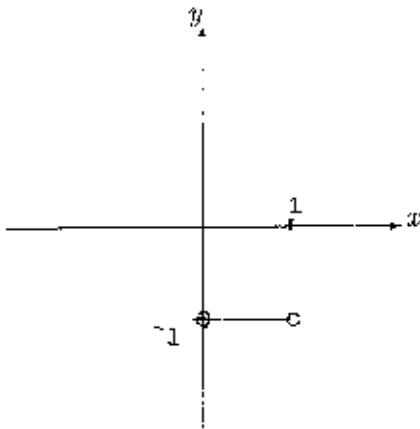
(f) Determine the symmetry of the function  $h(x)$ .

Even  $\hat{h}(\omega) \equiv \text{even} \Rightarrow h(x) \text{ is even}$

(g) Calculate the real Fourier series representation of  $g(x)$ .

$$g(x) = \sum_{n=-\infty}^{-1} \frac{i(-1)^n}{2\pi n} e^{inx} + \sum_{n=1}^{\infty} \frac{i(-1)^n}{2\pi n} e^{inx} = \sum_{n=1}^{\infty} -i \frac{(-1)^n}{2\pi n} e^{-inx} + \frac{i(-1)^n}{2\pi n} e^{inx} = \sum_{n=1}^{\infty} \frac{i(-1)^n}{2\pi n} \left[ e^{inx} - e^{-inx} \right] = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\pi n} \sin(nx)$$

3. (10 Points) Below is the graph of the function  $f$ .



(a) Find the Fourier cosine half-range expansion of  $f$ .

$$b_n = 0, \quad a_n = \frac{2}{1} \int_0^1 -1 \cos(n\pi x) dx = -2 \left[ \frac{\sin(n\pi x)}{n\pi} \right] \Big|_0^1 = 0$$

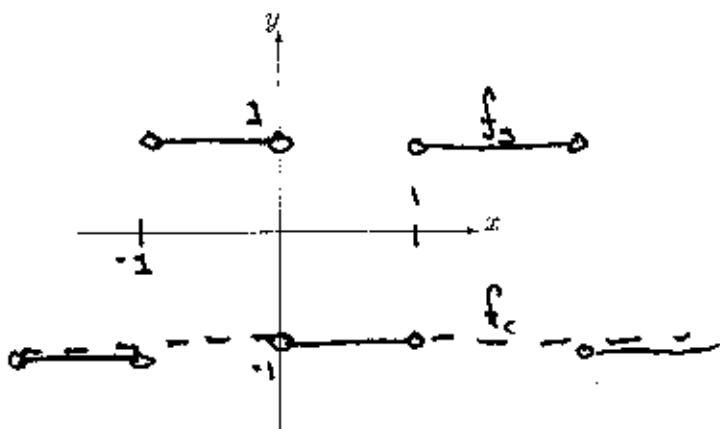
$$a_0 = \frac{2}{1} \int_0^1 -1 dx = -1 \left[ x \right] \Big|_0^1 = -1 \Rightarrow f_c(x) = 1$$

(b) Find the Fourier sine half-range expansion of  $f$ .

$$\begin{aligned} b_n &= \frac{2}{1} \int_0^1 -1 \sin(n\pi x) dx = -2 \left[ -\frac{\cos(n\pi x)}{n\pi} \right] \Big|_0^1 = \frac{2}{n\pi} [\cos(n\pi) - 1] = \\ &= \frac{2}{n\pi} [(-1)^n - 1] = b_n \end{aligned}$$

$$\Rightarrow f_s(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi} [(-1)^n - 1] \sin(nx)$$

(c) Graph the Fourier cosine half-range expansion of  $f$  with a dashed line and the Fourier sine half-range expansion of  $f$  with a solid line on the axes below.



4. (10 Points) Find the complex Fourier series representation of  $f(x) = \pi$ ,  $x \in (-\pi, \pi)$ , where  $f(x+2\pi) = f(x)$ .

$$C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \pi e^{-inx} dx = \frac{1}{2} \left[ -\frac{1}{in} e^{-inx} \right]_{-\pi}^{\pi} = \frac{-1}{2in} e^{-in\pi} - e^{in\pi} = 0$$

$$S = \frac{1}{2\pi} \int_{-\pi}^{\pi} \pi dx = \frac{\pi}{2\pi} \cdot 2\pi = \pi$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} = \pi$$

5. (10 Points) Suppose that  $f$  is given as,

$$f(x) = \begin{cases} x+1, & -1 \leq x < 0 \\ 1-x, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases} . \quad (1)$$

Calculate the complex Fourier transform of  $f$ . Noting the identity,  $2\sin^2(y) = 1 - \cos(2y)$ , simplify as much as possible.

$$\begin{aligned}
 \hat{f}(w) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ixw} dx = \frac{1}{\sqrt{2\pi}} \left[ \int_{-1}^0 (x+1) e^{-ixw} dx + \int_0^1 (1-x) e^{-ixw} dx \right] = \\
 &= \frac{1}{\sqrt{2\pi}} \left[ \int_{-1}^0 x e^{-ixw} dx + \int_{-1}^0 e^{-ixw} dx + \int_0^1 e^{-ixw} dx - \int_0^1 x e^{-ixw} dx \right] = \quad \begin{array}{l} u \\ x \\ x+1 \\ 1-x \\ 0 \end{array} \quad \begin{array}{l} du \\ e^{-ixw} \\ e^{-ixw} \\ e^{-ixw} \\ 1 \end{array} \\
 &= \frac{1}{\sqrt{2\pi}} \left[ \left. \frac{x e^{-ixw}}{-iw} \right|_{-1}^0 + \left. \frac{e^{-ixw}}{w^2} \right|_{-1}^0 + \left. \frac{1}{iw} e^{-ixw} \right|_0^1 + - \left[ \left. \frac{x e^{-ixw}}{iw} \right|_0^1 + \left. \frac{e^{-ixw}}{w^2} \right|_0^1 \right] \right] = \\
 &= \frac{1}{\sqrt{2\pi}} \left[ \cancel{\left. \frac{-e^{iwx}}{iw} \right|_{-1}^0} + \cancel{\left. \frac{1}{w^2} - \frac{e^{-iwx}}{w^2} \right|_{-1}^0} + \cancel{\left. \frac{1}{iw} e^{-iwx} \right|_0^1} + \cancel{\left[ \left. \frac{-xe^{-iwx}}{iw} \right|_0^1 + \left. \frac{e^{-iwx}}{w^2} \right|_0^1 \right]} \right] = \\
 &= \frac{1}{\sqrt{2\pi}} \left[ \cancel{\frac{2}{w^2}} - \cancel{\frac{2e^{iwx}}{w^2}} \right] = \frac{3}{\sqrt{2\pi}} \left[ \cancel{\frac{2}{w^2}} - \cancel{e^{iwx}} \right] = \frac{1}{\sqrt{2\pi}} \left[ \frac{2}{w^2} - \frac{(e^{iwx} + e^{-iwx})}{w^2} \right] = \\
 &= \frac{1}{\sqrt{2\pi}} \left[ \frac{2}{w^2} - \frac{2 \cos(w)}{w^2} \right] = \frac{2}{\sqrt{2\pi}} \frac{\sin^2(w/2)}{w^2} = \frac{2}{\sqrt{2\pi}} \frac{1 - \cos(w)}{w^2}
 \end{aligned}$$

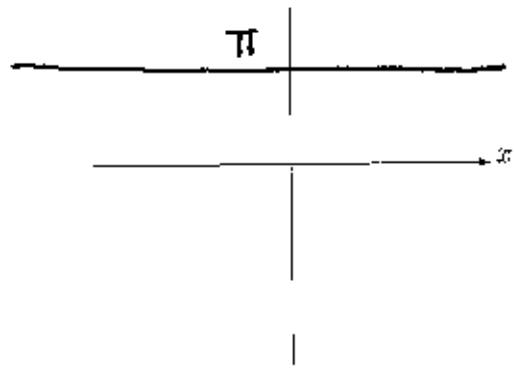
6. (Extra Credit)

(a) Let  $\hat{f}(\omega) = \sum_{n=-\infty}^{\infty} \sqrt{2\pi} c_n \delta(\omega - n)$ . Find the inverse Fourier transform of  $\hat{f}$ .

$$f(x) = \mathcal{F}^{-1}\{\hat{f}(\omega)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \sqrt{2\pi} c_n \delta(\omega - n) e^{inx} d\omega = \sum_{n=-\infty}^{\infty} c_n \int_{-\infty}^{\infty} \delta(\omega - n) e^{inx} d\omega =$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

(b) Suppose that  $c_n = \frac{e^{inx} - e^{-inx}}{n}$  for  $n \neq 0$  and  $c_0 = \pi$ . Graph the function  $f(x) = \mathcal{F}^{-1}\{\hat{f}\}$  below.



$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} = \pi$$