E. Kreyszig, Advanced Engineering Mathematics, 9^{th} ed.	Section 11.4, pgs. 47xx-4xx
Lecture: Fourier Series	Module: 10
Suggested Problem Set: {2, 9, 11 }	March 4, 2009

Quote of Lecture 10
Juliet: What's in a name? That which we call a rose by any other name would smell as sweet.
Shakespeare : Romeo and Juliet (1591)

1 Review

So, at this point we have the following,

$$
\begin{aligned}
f(x) &= a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right), & (1)\\
a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)dx, & (2)\\
a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx, & (3)\\
b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx, & (4)
\end{aligned}
$$

which defines the Fourier series and it's associated coefficients for a $2L$-periodic function, where L is a scaling parameter introduced to control the length of the period. We also have the following important results:

- Any function for, which the integrals (2)-(4) are defined has a Fourier series representation. Notice that this does not require the function to be periodic, but the Fourier series will induce this function to be periodic with principle domain (-L,L).
- The Fourier series may actually differ from the function $f(x)$ at a countably infinite amount of points. We can know where this might occur by knowing the jump-discontinuities of f and we have that the Fourier series will average the right and left hand limits at these points.
- The Fourier series represents the function f in terms of it's oscillatory features for which the data f supplies the amplitudes for each oscillatory mode. [1]
- The Fourier series represents the function f in terms of it's even components and odd components. [2]

2 Lecture Overview

Now we are going to make use of the well celebrated Euler's formula,

$$e^{i\theta} = \cos(\theta) + i\sin(\theta), \quad i = \sqrt{-1}, \qquad (5)$$

so that we can rewrite (1)-(4) in its complex form,

$$
\begin{aligned}
f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{-i\frac{n\pi}{L}x}, & (6)\\
c_n &= \frac{1}{2L}\int_{-L}^{L} f(x)e^{i\frac{n\pi}{L}x}dx, & (7)
\end{aligned}
$$

[1] Each term in the series is called a Fourier mode and the lowest order term is often called the Fundamental mode.

[2] If the function f has symmetry then the equations (1)-(4) simplify according to the intal properties of symmetric functions.

which is tidy but lacks some of the clarity of the real-form. [3] From this form one can always derive the real Fourier series form and moreover if the function f is symmetric then this immediately simplifies to a Fourier cosine or Fourier sine series. The following outlines some pros and cons:

Pro: We need only remember 2 formula instead of 4.

Pro: Integrations involving exponential functions greatly simplify.

Con: The case for when n is often a special case (notice that $c_0 = a_0$) where the coefficient becomes singular due to anti-differentiation of the exponential function.

Con: From the complex form the graph of the periodic function is not as accessible.

Lastly, to calculate the energy in a 'signal' we note that the energy of a sinusoid is proportional the square of it's amplitude [4] then we can conclude that the energy of a signal can be found by it's Fourier coefficients as

$$E \propto a_0^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2, \tag{9}$$

however in (6)-(7) the Fourier coefficients may be complex and the connection to energy is not as clear. In this case we have the following:

$$E \propto \sum_{n=-\infty}^{\infty} |c_n|^2, \tag{10}$$

where $|c_n|^2 = c_n \bar{c}_n$. [5]

3 Lecture Goals

Our goals with this material will be:

- Understand the connections both similarities and differences between complex and real Fourier series representations of functions.

4 Lecture Objectives

The objectives of these lessons will be:

- Derive the complex Fourier series using the real Fourier series and associated coefficients.
- Learn to convert the complex Fourier series into a real Fourier series through algebraic simplifications.

[3] The coefficients, which we derive from a_n and b_n in class, can also be derived from the following orthogonality relation:

$$\left\langle e^{-i\frac{n\pi}{L}}, e^{-i\frac{m\pi}{L}} \right\rangle = 2L\delta_{nm} \tag{8}$$

[4] http:/.glenbrook.k12.il.us/gbssci/phys/Class/waves/u10l2c.html

[5] Here the 'bar' denotes complex conjugation. If $z = \alpha + \beta i$ then $\bar{z} = \alpha - \beta i$ and one can easily conclude that $z\bar{z} \in \mathbb{R}$ as we would expect for a quantity like energy.