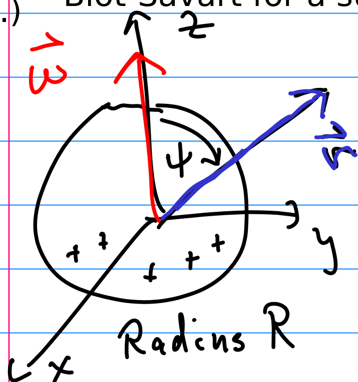
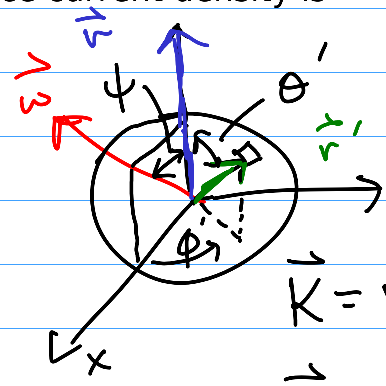
Hmwk 4

1.) Biot-Savart for a surface current density is

$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{\vec{K}(\vec{r}\,') \times \hat{\mathscr{r}}}{\mathscr{r}^2}\, da'$$




$$da' = R\sin\theta'\, d\phi'\, R\, d\theta'$$

$$\vec{K} = \sigma\vec{v} = \sigma\,\vec{\omega}\times\vec{r}\,'$$

$$\vec{K} = \sigma\omega\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \sin\psi & 0 & \cos\psi \\ R\sin\theta'\cos\phi' & R\sin\theta'\sin\phi' & R\cos\theta' \end{vmatrix}$$

Let $\vec{\omega}$ be in the x-z plane

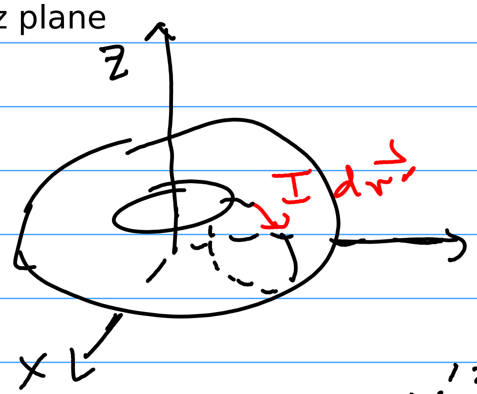
$$\vec{\omega} = \omega\sin\psi\,\hat{x} + \omega\cos\psi\,\hat{z}$$

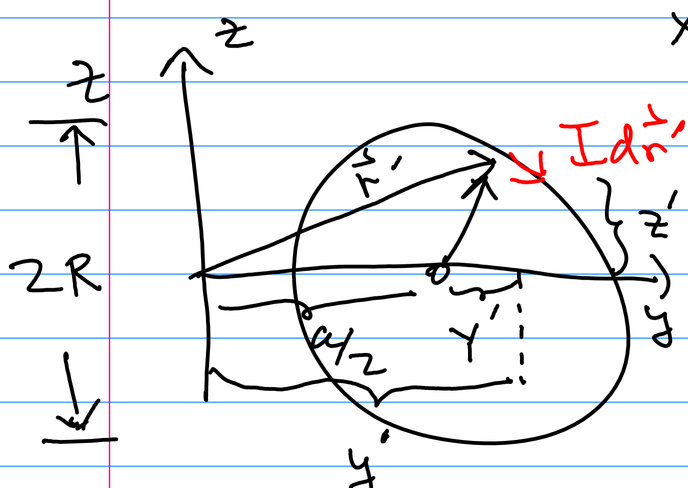
$$\vec{r}\,' = R\sin\theta'\cos\phi'\,\hat{x} + R\sin\theta'\sin\phi'\,\hat{y} + R\cos\theta'\,\hat{z}$$

2.) Start with the current in the y-z plane

$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{I\, d\vec{r}\,' \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$

Find $I\,d\vec{r}\,'$ first.




$$y'^2 + z'^2 = R^2$$

$$\vec{r}\,' = y'\,\hat{y} + z'\,\hat{z}$$

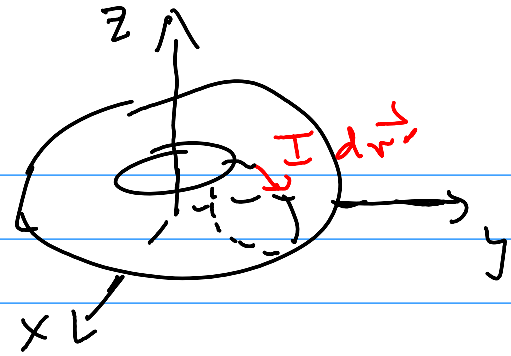
$$\vec{r}\,' = \left(\frac{a}{2} + Y'\right)\hat{y} + \sqrt{R^2 - Y'^2}\,\hat{z}$$

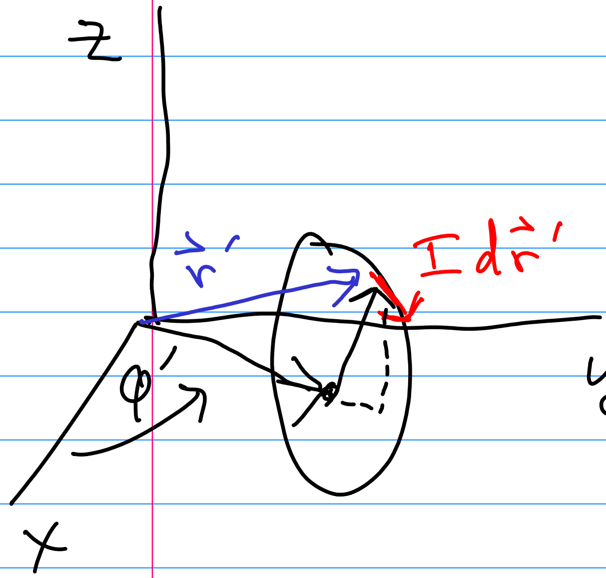
$$d\vec{r}\,' = dY'\,\hat{y} + \frac{1}{2}\,\frac{-2Y'\,dY'}{\sqrt{\phantom{x}}}\,\hat{z}$$

This looks like a multivalued function. However sign is needed for the lower half of circle.

$$\vec{r}\,' = \left(\frac{a}{2} + Y'\right)\hat{y} \pm \sqrt{R^2 - Y'^2}\,\hat{z}$$

Let the current element be in the first quadrant.




$$\vec{r}\,' = \left(\frac{a}{2} + Y'\right)\cos\varphi'\,\hat{x} + \left(\frac{a}{2} + Y'\right)\sin\varphi'\,\hat{y} + \sqrt{R^2 - Y'^2}\,\hat{z}$$

$$\delta f(x,y) = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y = df$$

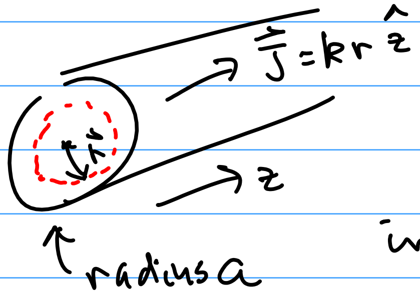
$$d\vec{r}\,' = \left[\left(\frac{a}{2} + Y'\right)(-\sin\varphi')\,d\varphi' + dY'\cos\varphi'\right]\hat{x} + \left[\left(\frac{a}{2} + Y'\right)(\cos\varphi')\,d\varphi' + dY'\sin\varphi'\right]\hat{y} + \frac{1}{2}\frac{(-2Y')}{\sqrt{\phantom{x}}}\,dY'\,\hat{z}$$

$$\vec{B} = \frac{\mu_0}{4\pi}\int\frac{I\,d\vec{r}\,' \times \hat{\mathscr{r}}}{\mathscr{r}^2} \qquad \vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

$\varphi'$ goes from 0 to $2\pi$

$Y'$ goes from $-R$ to $R$

3.) Ampere's law $\oint \vec{B}\cdot d\vec{\ell} = \mu_0 \int \vec{J}\cdot d\vec{a}$




From symmetry $B$ is in $\hat{\varphi}$

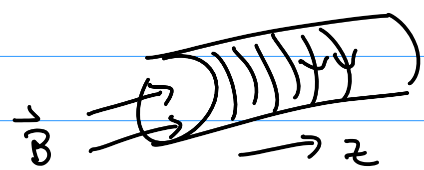
$\oint \vec{B}\cdot d\vec{\ell} = B\,2\pi r = \mu_0 \int_0^{2\pi}\int_0^{r_{in}} kr\hat{z}\cdot r\,d\varphi\,dr\,\hat{z}$

inside $r_{in} < a$ $B = \mu_0$

outside $r_{out} > a$ $B\,2\pi r_{out} = \mu_0 \int_0^{2\pi}\int_0^{a} kr\hat{z}\cdot r\,d\varphi\,dr\,\hat{z}$

limit on $r$ is $0 \to a$ since $J = 0$ from $a \to r_{out}$.

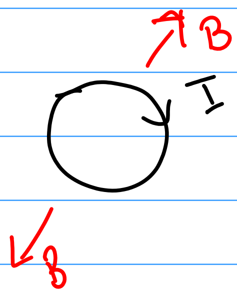
4.) Solenoid




B outside a solenoid is zero and is non zero inside, pointing along z.

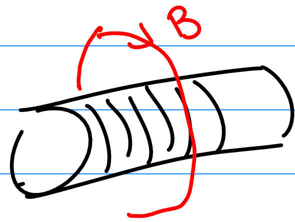
How do we deduce this direction for B?

Assume it is radial.

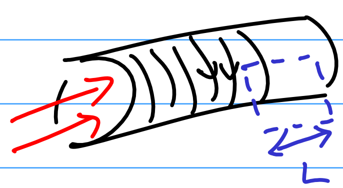



You can find symmetrically located I dr which cancel the radial component.

Assume it is azimuthal.




Then $\oint \vec{B}\cdot d\vec{r} \neq 0$ but there is no current enclosed by this path



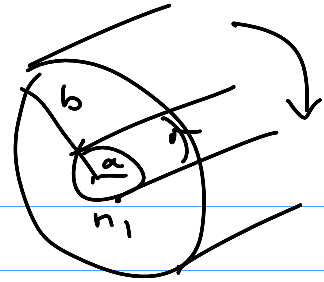

$\oint \vec{B}\cdot d\vec{r} = BL = \mu_0 I_{encl}$

$= \mu_0 n I L$

$n$: # wires / meter; $I$: Amps / wire; $L$: meters

$\vec{B} = \mu_0 n I \hat{z}$

2 solenoids



Use the superposition principle:

$\vec{B}_1 = -\mu_0 n_1 I \hat{z} \quad r < a$

$B_2 = \mu_0 n_2 I \hat{z} \quad r < b$

$(-\mu_0 n_1 I + \mu_0 n_2 I)\hat{z}$

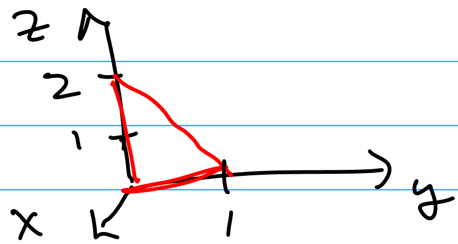
$\vec{B}(r<a) = \vec{B}_1 + \vec{B}_2$

$B(r>a) = B_1 + B_2 = 0$

5.) $\vec{\nabla} \times \vec{E} = 0$ Neither is an electrostatic field.

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2yz & 3zx \end{vmatrix} \neq 0 \neq \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & 2xy+z^2 & 2yz \end{vmatrix}$$

6.)



3 paths (a) $x = z = 0 \quad dx = dz = 0 \quad y: 0 \to 1$

$\vec{v} \cdot d\vec{r} = (y + 3x)dy = y\,dy \qquad \int \vec{v} \cdot d\vec{r} = \int_0^1 y\,dy = \frac{1}{2}$

(b) $x = 0; z = 2 - 2y; \; dz = -2dy; \; y: 1 \to 0$

$\vec{v} \cdot d\vec{r} = (y + 3x)dy + 6dz = y\,dy - 12dy$

$\int \vec{v} \cdot d\vec{r} = \int_1^0 (y - 12)dy = -\frac{1}{2} + 12$

(c) $x = y = 0 \quad dx = dy = 0 \quad z: 2 \to 0 \qquad \vec{v} \cdot d\vec{r} = 6dz$

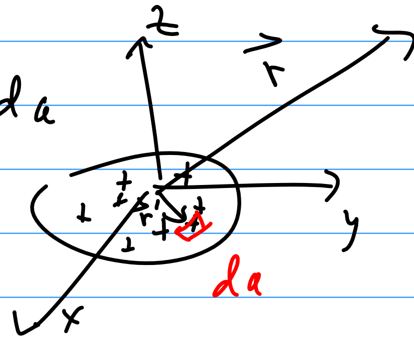
$\int \vec{v} \cdot d\vec{r} = \int_2^0 6dz = -12$

$$\oint \vec{v}\cdot d\vec{r} = \frac{1}{2} - \frac{1}{2} + 12 - 12 = 0$$

$$\int \vec{\nabla}\times\vec{v}\cdot d\vec{a} \qquad d\vec{a} = dy\,dz\,\hat{x}$$

$(\vec{\nabla}\times\vec{v})_x$ is all that is needed & it $= 0$.

7.) $dV = \dfrac{k\,dQ}{r}$ $\qquad dQ = \sigma\,da$




$da = r'\,d\varphi'\,dr'$

$\vec{r}\,' = r'\cos\varphi'\,\hat{x} + r'\sin\varphi'\,\hat{y} \qquad \vec{r} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$

$$|\vec{r}| = \sqrt{(x - r'\cos\varphi')^2 + (y - r'\sin\varphi')^2 + z^2}$$

$r' : 0 \to R \qquad \varphi' : 0 \to 2\pi$

8.) Take minus the gradient of the result. Note that if you had determined the potential only on the z axis you would not have the x and y dependence needed to take the gradient and therefore you would not be able to find E everywhere from this gradient.