tem of ordinary differential equations with a delay, called a system of delay-differential equations.

Integrating equation 64.2, yields

$$\frac{dx_n(t+T)}{dt} = -\lambda(x_n(t) - x_{n-1}(t)) + d_n,$$

an equation relating the velocity of cars at a later time to the distance between cars. Imagine a **steady-state** situation in which all cars are equidistant apart, and hence moving at the same velocity. Thus

$$\frac{dx_n(t+T)}{dt} = \frac{dx_n(t)}{dt},$$

and hence letting $d_n = d$

$$\frac{dx_n(t)}{dt} = -\lambda(x_n(t) - x_{n-1}(t)) + d.$$

Since

$$\boxed{x_{n-1}(t) - x_n(t) = \frac{1}{\rho}} \tag{64.3}$$

is a reasonable definition of traffic density (see equation 58.1), this car-following model yields a velocity-density relationship

$$u = \frac{\lambda}{\rho} + d.$$

We choose the one arbitrary constant d, such that at maximum density (bumper-to-bumper traffic) $u = 0$. In other words,

$$0 = \frac{\lambda}{\rho_{max}} + d.$$

In this way the following velocity-density relationship is derived,

$$u = \lambda\left(\frac{1}{\rho} - \frac{1}{\rho_{max}}\right), \tag{64.4}$$

sketched in Fig. 64-1. How does this compare with experimental observations of velocity-density relationships? Equation 64.4 appears reasonable for large densities, i.e., near $\rho = \rho_{max}$. However, it predicts an infinite velocity at zero

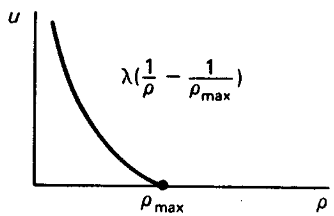

Figure 64-1 Steady-state car-following model: velocity-density relationship.

density. We can eliminate this problem, by noting that this model is not appropriate for small densities for the following reasons. At small densities, the changes in speed of a car are not due to the car in front. Instead it is more likely that the speed limit influences a car's velocity (and acceleration) at small densities. Thus we can hypothesize that equation 64.4 is valid only for large densities. For small densities, perhaps u is only limited by the speed limit, $u = u_{max}$. Thus

$$u = \begin{cases} u_{max} & \rho < \rho_c \\ \lambda\left(\frac{1}{\rho} - \frac{1}{\rho_{max}}\right) & \rho > \rho_c. \end{cases}$$

We choose the critical density ρ_c such that the velocity is a continuous function of density, as shown in Fig. 64-2. The flow $q = \rho u$ is thus

$$q = \begin{cases} u_{max}\rho & \rho < \rho_c \\ \lambda\left(1 - \frac{\rho}{\rho_{max}}\right) & \rho > \rho_c. \end{cases}$$

A comparison with the experimental data of Sec. 62 shows only a moderate agreement.

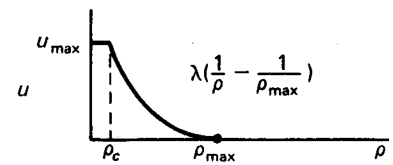

Figure 64-2 Speed limit cut-off of car-following velocity-density curve.

Let us attempt one improvement. Imagine two drivers each driving 20 m.p.h. (32 k.p.h.) faster than the car ahead, one 500 feet (160 meters) behind but the other only 25 feet (8 meters) behind. We know the drivers will decelerate quite differently. A driver's acceleration or deceleration also depends on the distance to the preceding car. The closer the driver is, the more likely the driver is to respond strongly to an observed relative velocity. The simplest way to model this is to let the sensitivity be inversely proportional to the distance

$$\lambda = \frac{c}{x_{n-1}(t) - x_n(t)}.$$

Thus the revised car-following model is

$$\frac{d^2x_n(t+T)}{dt^2} = c\,\frac{\frac{dx_n(t)}{dt} - \frac{dx_{n-1}(t)}{dt}}{x_n(t) - x_{n-1}(t)}, \tag{64.5}$$

a nonlinear car-following model, as opposed to equation 64.2 which is linear.