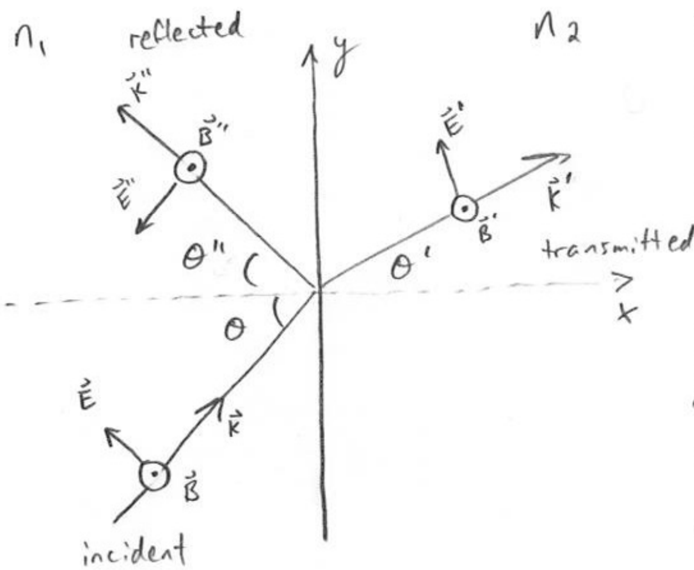
Recitation 2 – Dielectric interfaces

Let's suppose we have TM-polarized light incident on the boundary between two dielectrics with indices n_1 and n_2. The incident light makes some nonzero angle θ with the optical axis. Sketch that situation, including the directions of the k vector, E-field, and B-field for each of the three waves involved.



I'm not yet making any assumptions about θ, θ', and θ'', but we have shown in the past that $\theta = \theta''$ and $n_1 \sin\theta = n_2 \sin\theta'$.

Note that I'm fixing the relative directions of $\vec{k}$, $\bar{E}$, and $\bar{B}$ by using the Poynting vector. We know $\vec{k}$ is in the $\bar{E} \times \bar{B}$ direction.

Given an incident electric field of the form

$$\vec{E}_I(\vec{x}, t) = \vec{E}_0 e^{i(\vec{k}\cdot\vec{x} - \omega t + \delta)}$$

What phase angle δ could we choose to represent an incident E-field that has zero magnitude when $\vec{x}$ and t are zero?

Well, when $\vec{x}$ and t are zero, we get

$$\vec{E}_I = \vec{E}_0 e^{i\delta} \qquad \text{And} \quad e^{i(0)} = 1,$$

$$e^{i(\pi/2)} = i$$

$e^{i\delta}$ is never zero by itself for any real δ, but since we take the real part of these expressions to get the physical fields, and $\mathrm{Re}\{e^{i\pi/2}\} = 0$, a phase angle of $\delta = \pi/2$ would do it.

Write the four field boundary conditions that we can most easily apply to dielectric interface problems. Keep them in a general form; don't adapt them to this specific problem yet.

$D_{1\perp} = D_{2\perp}$, or $\varepsilon_1 E_{1,\perp} = \varepsilon_2 E_{2,\perp}$

$B_{1,\perp} = B_{2,\perp}$

$E_{1,\parallel} = E_{2,\parallel}$

$H_{1,\parallel} = H_{2,\parallel}$, or $B_{1,\parallel}/\mu_1 = B_{2,\parallel}/\mu_2$

These come from our Maxwell equations in matter with the free sources set to zero:

$$\nabla \cdot \vec{D} = 0 \qquad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \nabla \times \vec{H} = +\frac{\partial D}{\partial t}$$

Explain why we don't directly use the boundary condition that reads $E_{1,\perp} - E_{2,\perp} = \sigma/\varepsilon_0$.

The σ in that equation includes a bound charge that comes about through polarization. Since the polarization of the material depends on the fields that we're trying to solve for, that boundary condition ends up being inconvenient at best.

Explain why we can be sure that the incident, reflected, and transmitted fields will all have the same frequency ω. Be specific. Also tell me whether there's one particular boundary condition that contains this information, or if more than one has it.

All of the boundary conditions involve statements of the form (some fields) = (some other fields). And all of our fields have some kind of $e^{-i\omega t}$ time dependence. So any of our boundary conditions will generate an equation of the form

$$\left(\begin{matrix}\text{stuff that is}\\ t\text{-independent}\end{matrix}\right) e^{-i\omega t} + \left(\begin{matrix}\text{other}\\ t\text{-independent}\\ \text{stuff}\end{matrix}\right) e^{-i\omega'' t} = \left(\begin{matrix}\text{more}\\ t\text{-independent}\\ \text{stuff}\end{matrix}\right) e^{-i\omega' t}$$

And such equations can only be satisfied for all t if

$$\omega = \omega' = \omega''.$$

How, specifically, is the wavelength in medium 2 related to the wavelength in medium 1, and how do you know?

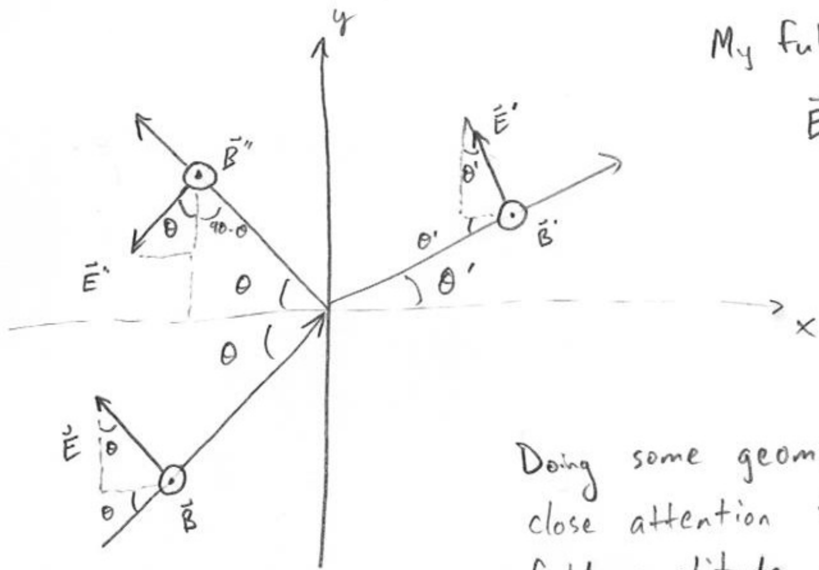
We can use Maxwell's equations and the basic form of a plane wave to show that $v = \lambda f$ quite generally, and since $\omega = \omega' = \omega''$, so must $f = f' = f''$.

With that in mind, fields in medium 1 satisfy $\frac{c}{n_1} = \lambda_1 f$

And fields in medium 2 satisfy $\frac{c}{n_2} = \lambda_2 f$

Thus $\frac{c}{n_1 \lambda_1} = f = \frac{c}{n_2 \lambda_2}$, and $\frac{1}{n_1 \lambda_1} = \frac{1}{n_2 \lambda_2}$, and $\boxed{\lambda_2 = \frac{n_1}{n_2} \lambda_1}$

Write full equations for the incident, reflected, and transmitted E-fields. Then use each boundary condition to generate an equation that relates the amplitudes of those fields. Get things simplified to the point that there's nothing left to do but solve for the field amplitude ratios, and write things in terms of the given indices of refraction. You can set the arbitrary phase angle δ to zero for simplicity.



My full $\vec{E}$-fields look like:

$$\vec{E}_{incident} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

$$\vec{E}_{reflected} = \vec{E}_0'' e^{i(\vec{k}''\cdot\vec{x} - \omega t)}$$

$$\vec{E}_{transmitted} = \vec{E}_0' e^{i(\vec{k}'\cdot\vec{x} - \omega t)}$$

Doing some geometry (see diagram) and paying close attention to signage, I can write my field amplitude vectors as:

$$\vec{E}_0 = -E_0 \sin\theta\, \hat{\imath} + E_0 \cos\theta\, \hat{\jmath}$$

$$\vec{E}_0'' = -E_0'' \sin\theta\, \hat{\imath} - E_0'' \cos\theta\, \hat{\jmath}$$

$$\vec{E}_0' = -E_0' \sin\theta'\, \hat{\imath} + E_0' \cos\theta'\, \hat{\jmath}$$

I know all the $e^{-i\omega t}$ terms will cancel when I go to set up boundary condition equations. And $e^{i\vec{k}\cdot\vec{x}} = e^{i(k_x x + k_y y + k_z z)}$. Letting the interface be at $x=0$ kills the $e^{ik_x x}$ piece. Letting the our $\vec{k}$ vectors sit entirely in the xy plane (so $k_z = 0$) kills the $e^{ik_z z}$ term, and the fact that our boundary conditions must hold at all y (plane waves are infinite in extent) makes $k_y = k_y' = k_y''$, so all the $e^{ik_y y}$ terms cancel. Thus, our boundary condition equations will end up being entirely in terms of field amplitudes.

We'll go down the list:

$\epsilon_1 E_{1_\perp} = \epsilon_2 E_{2_\perp}$ yields $-\epsilon_1 E_0 \sin\theta - \epsilon_1 E_0'' \sin\theta = -\epsilon_2 E_0' \sin\theta'$

$$\Rightarrow \quad \epsilon_1 \sin\theta (E_0 + E_0'') = \epsilon_2 \sin\theta' E_0'$$

I'm inclined to simplify this immediately, and we know $n_1 \sin\theta = n_2 \sin\theta'$

$\Rightarrow \sin\theta' = \frac{n_1}{n_2}\sin\theta$

$$\Rightarrow \quad \epsilon_1 \cancel{\sin\theta} (E_0 + E_0'') = \epsilon_2 \frac{n_1}{n_2} \cancel{\sin\theta} E_0'$$

And we wanted things in terms of n_1 and n_2, so let's figure out how to substitute for ϵ:

$$\frac{1}{\sqrt{\epsilon\mu}} = v, \quad \epsilon\mu = \frac{1}{v^2}, \quad c^2\epsilon\mu = \left(\frac{c}{v}\right)^2 = n^2$$

$$\Rightarrow \quad \epsilon = \frac{n^2}{c^2\mu}$$

Substituting,

$$\frac{n_1^2}{\cancel{c^2\mu_1}}(E_0 + E_0'') = \frac{n_2^2}{\cancel{c^2\mu_2}}\frac{n_1}{n_2} E_0' \qquad \text{And } \mu_1 \approx \mu_2$$

$$\Rightarrow \quad \boxed{n_1 (E_0 + E_0'') = n_2 E_0'}$$

That's one. We need two. Let's use the condition on $E_\parallel$:

$E_{1_\parallel} = E_{2_\parallel} \quad \Rightarrow \quad E_0 \cos\theta - E_0'' \cos\theta = E_0' \cos\theta'$

$$\Rightarrow \quad \boxed{(E_0 - E_0'') \cos\theta = E_0' \cos\theta'}$$

And that's about as simple as it gets. We now have two independent equations and two unknowns, so we're ready to go.

If you have time, go ahead and solve for the ratios $\frac{E_0'}{E_0}$ and $\frac{E_0''}{E_0}$ to obtain two of the Fresnel equations.

Two equations, two unknowns. Not so bad. Starting from the first equation:

$$n_1\left(E_0 + E_0''\right) = n_2 E_0' \quad \Rightarrow \quad \frac{n_1}{n_2}\left(\frac{E_0}{E_0} + \frac{E_0''}{E_0}\right) = \frac{E_0'}{E_0} \quad \Rightarrow \quad \boxed{\frac{E_0'}{E_0} = \frac{n_1}{n_2} + \frac{n_1}{n_2}\frac{E_0''}{E_0}} \qquad (1)$$

And we can substitute this into the second equation:

$$\left(E_0 - E_0''\right)\cos\theta = E_0'\cos\theta' \Rightarrow \frac{E_0}{E_0} - \frac{E_0''}{E_0} = \frac{E_0'}{E_0}\frac{\cos\theta'}{\cos\theta} \Rightarrow 1 - \frac{E_0''}{E_0} = \frac{E_0'}{E_0}\frac{\cos\theta'}{\cos\theta}$$

$$\Rightarrow 1 - \frac{E_0''}{E_0} = \left[\frac{n_1}{n_2} + \frac{n_1}{n_2}\frac{E_0''}{E_0}\right]\frac{\cos\theta'}{\cos\theta} \Rightarrow 1 - \frac{n_1}{n_2}\frac{\cos\theta'}{\cos\theta} = \frac{n_1\cos\theta'}{n_2\cos\theta}\frac{E_0''}{E_0} + \frac{E_0''}{E_0}$$

$$\Rightarrow 1 - \frac{n_1\cos\theta'}{n_2\cos\theta} = \left(1 + \frac{n_1}{n_2}\frac{\cos\theta'}{\cos\theta}\right)\frac{E_0''}{E_0} \Rightarrow \frac{E_0''}{E_0} = \frac{1 - \frac{n_1\cos\theta'}{n_2\cos\theta}}{1 + \frac{n_1\cos\theta'}{n_2\cos\theta}} \Rightarrow \boxed{\frac{E_0''}{E_0} = \frac{n_2\cos\theta - n_1\cos\theta'}{n_2\cos\theta - n_1\cos\theta'}}$$

That's one Fresnel equation. We can back-substitute into (1) to get the other:

$$\frac{E_0'}{E_0} = \frac{n_1}{n_2} + \frac{n_1}{n_2}\left[\frac{n_2\cos\theta - n_1\cos\theta'}{n_2\cos\theta + n_1\cos\theta'}\right] = \frac{n_1}{n_2}\left[\frac{n_2\cos\theta + n_1\cos\theta'}{n_2\cos\theta + n_1\cos\theta'} + \frac{n_2\cos\theta - n_1\cos\theta'}{n_2\cos\theta + n_1\cos\theta'}\right]$$

$$= \frac{n_1}{\cancel{n_2}}\left[\frac{2\cancel{n_2}\cos\theta}{n_2\cos\theta + n_1\cos\theta'}\right] \Rightarrow \boxed{\frac{E_0'}{E_0} = \frac{2n_1\cos\theta}{n_2\cos\theta + n_1\cos\theta'}}$$

And sometimes these get recast in other forms, but that's a good starting point.