MATH 225 - May 22, 2008
Exam I - 50 minutes - 50 Points

NAME: Key
PROBLEMS DO NOT HAVE EQUAL WEIGHT

In order to receive full credit, SHOW ALL YOUR WORK. Full credit will be given only if all reasoning and work is provided. When applicable, please enclose your final answers in boxes.

1. (10 Points) Show that the following functions are solutions to their corresponding ODE's.

   (a) The function $y(t) = t^2 + t^{-2}$ for the differential equation $t\dfrac{dy}{dt} + 2y = 4t^2$

   $\Rightarrow y'(t) = 2t - 2t^{-3} \Rightarrow t\dfrac{dy}{dt} + 2y = t(2t - 2t^{-3}) + 2(2t - 2t^{-3}) = 4t^2$ ✓

   (b) The function $y(t) = c_1 \cos(3t) + c_2 \sin(3t)$ where $c_1, c_2 \in \mathbb{R}$ for the differential equation $\dfrac{d^2y}{dt^2} + 9y = 0$

   $\Rightarrow y''(t) = -3^2 c_1 \cos(3t) + -3^2 c_2 \sin(3t) = -9y(t) \Rightarrow y'' + 9y = -9y + 9y = 0$ ✓

2. (5 Points) Consider the initial value problem (IVP),

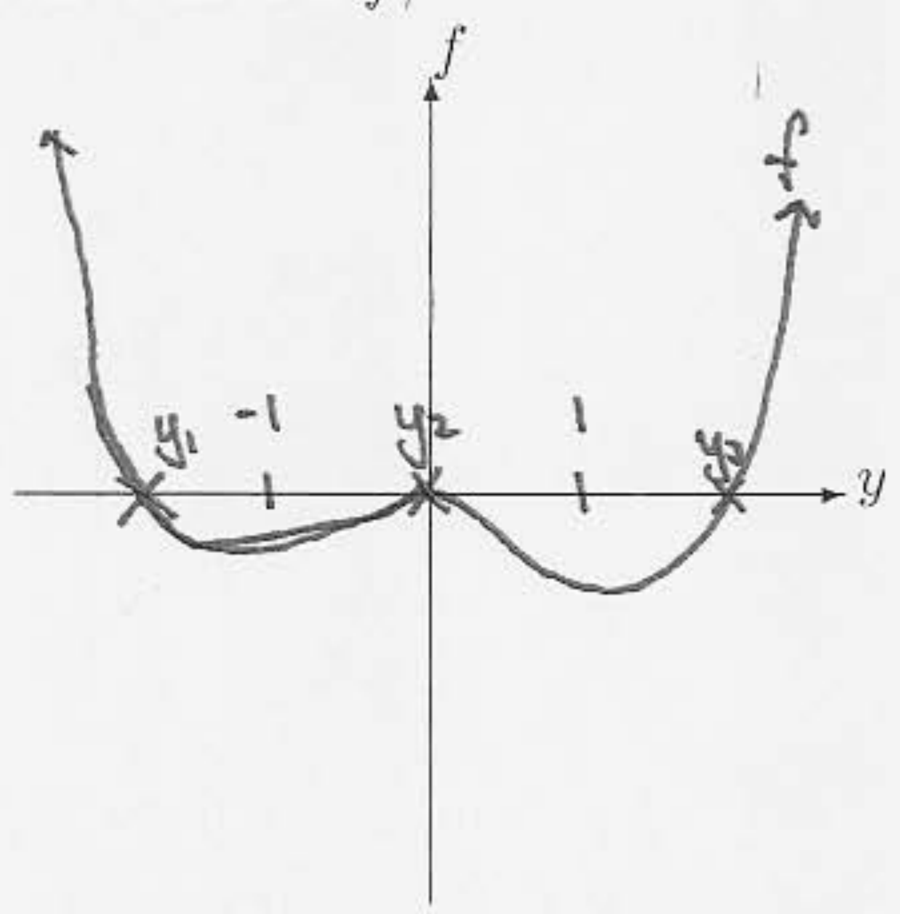
$$\frac{dy}{dt} = 2y - 1, \quad y(0) = 3. \qquad (1)$$

Use Euler's Method with a step size of 1 on the interval $0 \le t \le 3$ to approximate the solution to the initial value problem.

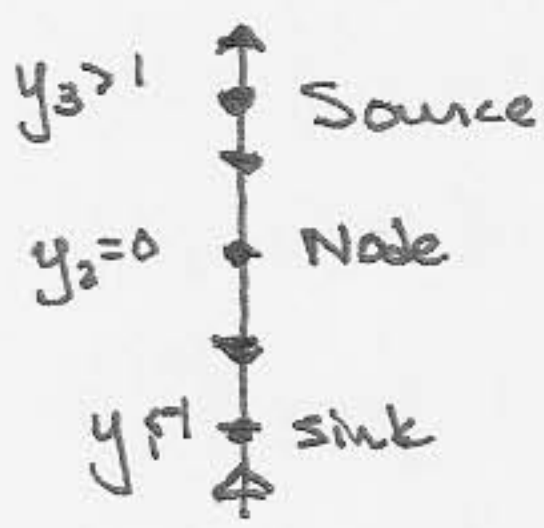
$y_{k+1} = \Delta t\, f(t_k, y_k) + y_k$

k	t_k	y_k	$f(t_k, y_k)$
0	0	3	$2(3)-1 = 5$
1	1	8	15
2	2	23	31
3	3	54	Not Needed

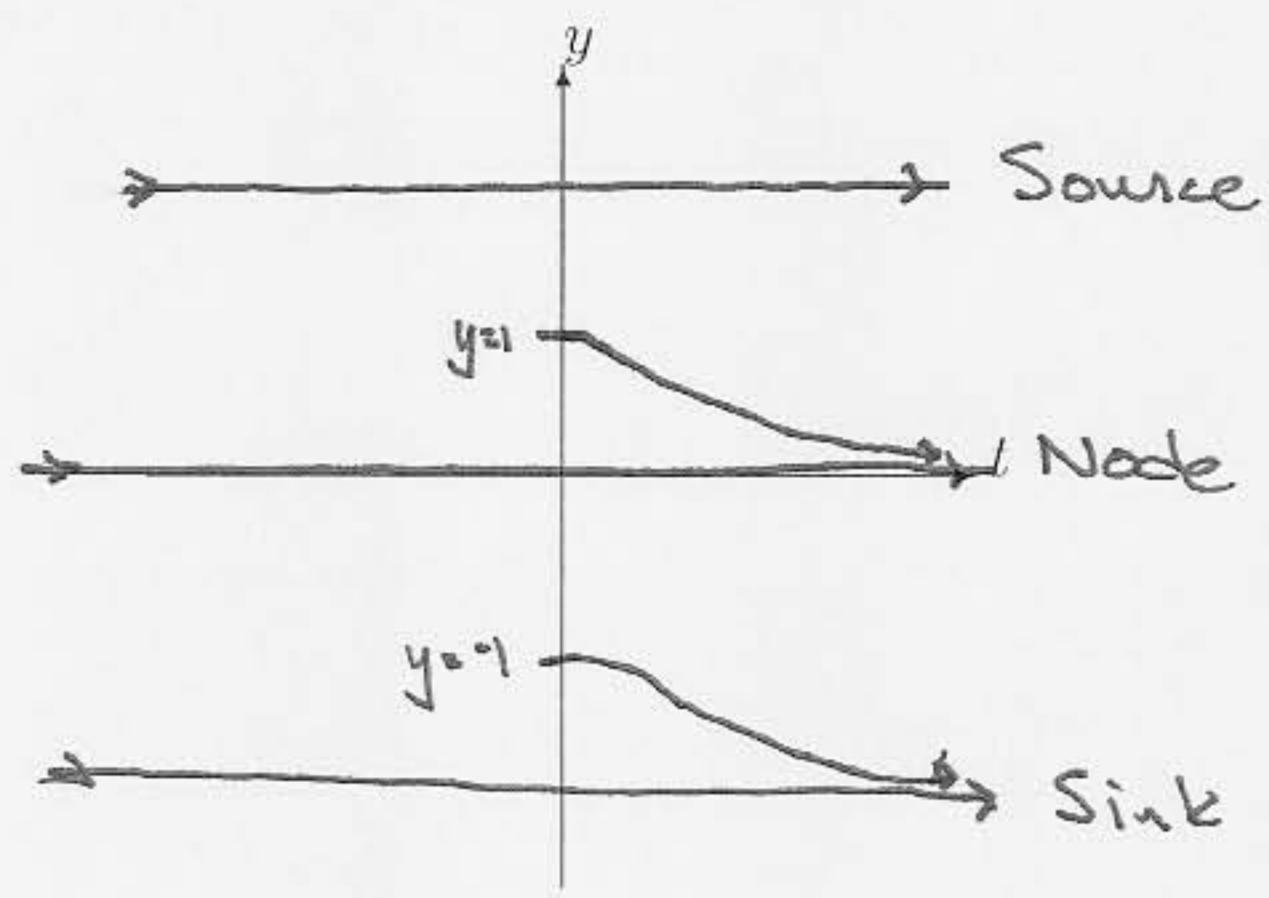
3. (10 Points) Given the graph of the function f,



(a) Sketch the phase line of $\dfrac{dy}{dt} = f(y)$ and label the equilibrium points as sinks, sources or nodes.



(b) Sketch the graphs of the solutions satisfying the initial conditions $y(0) = 0$, $y(0) = 1$, and $y(0) = -1$.



4. (5 Points) Given,

$$\frac{dy}{dt} = -y^2 + y + 2yt^2 + 2t - t^2 - t^4. \tag{2}$$

It can be shown that $y_1(t) = t^2$ and $y_2(t) = t^2 + 1$ are solutions to the differential equation. Is it true that if $y(t)$ is a solution to the differential equation and if $0 < y(0) < 1$ then $t^2 < y(t) < t^2 + 1$ for all t? Why or why not?

First Note that

f, and $\dfrac{\partial f}{\partial y}$ are continuous for all t, y

and that $0 = y_1(0) < y(0) < y_2(0) = 1 \underset{\text{By E+U Theorem.}}{\Rightarrow} y_1(t) < y(t) < y_2(t)$ for all t.

5. Find explicit solutions to the following differential equations. When initial conditions are given determine the value of the integration constant.

(a) $2\frac{dy}{dt} = y^2 - 1$ $\iff \int \frac{2\,dy}{y^2-1} = \int \left(\frac{+1}{y-1} + \frac{-1}{y+1} \right) dy = \cancel{\ln|y-1| - \ln|y+1|}\ \ln\left|\frac{y-1}{y+1}\right| = \int dt = t + C$

$$\Rightarrow y(t) = \frac{Ce^t + 1}{1 - Ce^t}$$

(b) $\frac{dy}{dt} = te^{-2t+3y}$ $\iff \int e^{-3y}\,dy = -\frac{1}{3}e^{-3y} = \int te^{-2t}\,dt = -\frac{t}{2}e^{-2t} - \frac{1}{4}e^{-2t} + C$

u	dv
t +	e^{-2t}
1 −	$-\frac{1}{2}e^{-2t}$
0	$\frac{1}{4}e^{-2t}$

$$\iff e^{-3y} = \frac{3t}{2}e^{-2t} + \frac{3}{4}e^{-2t} + c$$

$$\Rightarrow y(t) = -\frac{1}{3}\ln\left(\frac{3}{2}te^{-2t} + \frac{3}{4}e^{-2t} + c\right)$$

(c) $y' + y = \cos(2t), \quad y(0) = 0$

$y_h(t) = e^{-t}$

$y_p(t) = A\cos(2t) + B\sin(2t)$

$\Rightarrow y_p' + y = -2A\sin(2t) + 2B\cos(2t) + A\cos(2t) + B\sin(2t) = \cos(2t)$

$\Rightarrow -2A + B = 0 \qquad 2B + A = 1$

$B = 2A \qquad \Rightarrow 2(2A) + A = 1 \Rightarrow A = \frac{1}{5} \Rightarrow B = \frac{2}{5}$

$y(t) = Ke^{-t} + \frac{1}{5}\cos(2t) + \frac{2}{5}\sin(2t) \Rightarrow y(0) = 0 = K + \frac{1}{5} \Rightarrow y(t) = -\frac{1}{5}e^{-t} + \frac{1}{5}\cos(2t) + \frac{2}{5}\sin(2t)$

(d) $y' = y + 5e^t$

$y_h = e^t$

$y_p(t) = Ate^t$

$y_p'(t) = Ae^t + Ate^t$

$\Rightarrow y' = Ae^t + Ate^t = Ate^t + 5e^t \Rightarrow A = 5 \Rightarrow y(t) = Ke^t + 5te^t$