Heat Eq$\underline{n}$ in $\mathbb{R}^{1+1}$ on a bounded domain with source term.

(1) $$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} + F(x,t), \quad x \in (0,L), \; t \in (0,\infty)$$

(2) $$u(0,t) = 0, \quad u(L,t) = 0,$$

(3) $$u(x,0) = \cancel{f(x)}\, g(x)$$

Notes:

- (2) implies the object is in contact with a $0°$ heat bath on (universe) a Relative scale

The homogeneous problem implies a sol$\underline{n}$ of the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(\sqrt{\lambda_n}\,x)\, e^{-c^2 \lambda_n t}, \quad \sqrt{\lambda_n} = \frac{n\pi}{L},$$

Which leads us to assume a sol$\underline{n}$ to (1) of the form,

$$u(x,t) = \sum_{n=1}^{\infty} \sin(\sqrt{\lambda_n}\,x)\, G_n(t)$$

Where $G_n(t)$ is an unknown dynamic. Under this assumptio we get

$$\frac{\partial u}{\partial t} - c^2 \frac{\partial^2 u}{\partial x^2} - F(x,t) = \sum_{n=1}^{\infty} \left[G_n' + c^2 \lambda_n G_n \right] \sin(\sqrt{\lambda_n}\,x) - F(x,t) = $$

Now assuming $F(x,t)$ has a sine half-Range Expansion gives

$$F(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin(\sqrt{\lambda_n} x) \, , \quad f_n(t) = \frac{2}{L}\int_0^L F(x,t)\sin(\sqrt{\lambda_n} x)\,dx$$

which means (1) becomes

$$\frac{\partial u}{\partial t} - c^2 \frac{\partial^2 u}{\partial x^2} - F(x,t) = \sum_{n=1}^{\infty} \sin(\sqrt{\lambda_n} x)\left[G_n' + (c\sqrt{\lambda_n})^2 G_n - f_n\right] = 0$$

Since these sine fn are orthogonal they are linearly independent. Thus, the only way to satisfy the previous Eqn is by forcing

$$G_n'(t) + c^2\lambda_n G_n(t) = f_n(t), \quad \text{for } n = 1, 2, 3, \ldots$$

which is a first order linear ODE that gives dynamics consistent with (1).

For Example, if

$$F(x,t) = e^{-t}\sin\left(\frac{2\pi}{L}x\right)$$

and

$$g(x) = \begin{cases} \frac{2k}{L}x \, , & x \in (0, L/2) \\ \frac{2k}{L}(L-x), & x \in (\frac{L}{2}, L) \end{cases}$$

We get

$$f_n(t) = \frac{2}{L}\int_0^L e^{-t}\sin\left(\frac{2\pi}{L}x\right)\sin\left(\sqrt{\lambda_n}x\right)dx =$$

$$= \frac{e^{-t}}{L}\int_{-L}^{L}\sin\left(\frac{2\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)dx = \frac{e^{-t}\cdot L\cdot}{L}\delta_{2n} =$$

$$= e^{-t}\delta_{2n}$$

$$\Rightarrow \quad G_n' + c^2\lambda_n G_n = e^{-t}\delta_{2n}\ , \quad n = 1, 2, 3, \ldots$$

For $n \neq 2$ the sol<u>n</u> is

$$G_n(t) = B_n e^{-c^2\lambda_n t}\ , \qquad n \neq 2$$

For $n = 2$ we have $G_n^h(t) = B_n e^{-c^2\lambda_n t}$

and $G_n^p(t) = Ae^{-t} \Rightarrow \dot{G}_n^p + c^2\lambda_n G_n = (-A + c^2\lambda_n A)e^{-t} = e^{-t}$

$$\Rightarrow \quad A = \frac{1}{c^2\lambda_n - 1}\ , \quad \underbrace{c^2\lambda_n \neq 1 \text{ for } n=2}_{}$$

I'm fine assuming this. Odds are the diffusivity c^2 is not $\frac{L^2}{4\pi^2}$.

That is $c^2 \neq \frac{1}{\lambda_2} = \frac{L^2}{2^2\pi^2}$

In the Event that is does then $G_p(t) = Ate^{-t}$

$$\Rightarrow$$

$$G_2(t) = B_2 e^{-c^2\lambda_2 t} + \frac{e^{-t}}{c^2\lambda_2 - 1}$$

Thus,

$$u(x,t) = \left[B_2 e^{-c^2\lambda_2 t} + \frac{e^{-t}}{c^2\lambda_2 - 1} \right] \sin\left(\frac{2\pi}{L}x\right) + $$

$$+ \sum_{\substack{n=1 \\ n\neq 2}}^{\infty} B_n e^{-c^2\lambda_n t} \sin\left(\frac{n\pi}{L}x\right)$$

is the general sol$^{\underline{n}}$. Application of the initial condition gives

$$g(x) = \left[B_2 + \frac{1}{c^2\lambda_2 - 1} \right] \sin\left(\frac{2\pi}{L}x\right) + \sum_{\substack{n=1 \\ n\neq 2}}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)$$

$\Rightarrow$ for $n \neq 2$

$$B_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)dx = \frac{2k}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

Based on a Result from Class

for $n = 2$

$$B_2 + \frac{1}{c^2\lambda_2 - 1} = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{2\pi}{L}x\right)dx = 0$$

$\Rightarrow$ $B_2 = \dfrac{-1}{c^2\lambda_2 - 1}$, which solves the initial value problem.