The 1D Wave Eq$\underline{n}$ on $\mathbb{R}^{+1}$!

9/19/12

When considering $u_{tt} = c^2 u_{xx}$ on $x \in (-\infty, \infty)$ the problem becomes more difficult b/c the domain leaves nothing to periodically Extend into.

See 9.17.12

Key Point 1: The wave Eq$\underline{n}$ on $\mathbb{R}^{+1}$ admits the general sol$\underline{n}$

$$u(x,t) = f(x-ct) + g(x+ct)$$

which can be readily verified.

Let $z_{\mp} = x \mp ct$ then

$$\frac{\partial u}{\partial t} = \frac{\partial f(z_-)}{\partial t} + \frac{\partial g(z_+)}{\partial t} =$$

$$= \frac{\partial z_-}{\partial t}\frac{\partial f}{\partial z_-} + \frac{\partial z_+}{\partial t}\frac{\partial g}{\partial z_+} = -c\frac{df}{dz_-} + c\frac{dg}{dz_+}$$

$$= -cf' + cg'$$

by a similar argument we get the relations:

$$\frac{\partial^2 u}{\partial t^2} = (-c)(-c)f'' + c \cdot c \cdot g''$$

$$\frac{\partial^2 u}{\partial x^2} = f'' + g''$$

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

For all x,t and fn f,g s.t. f'', g'' Exist.

Key Outcome: There ~~is~~ Exist soln to $u_{tt} = c^2 u_{xx}$ which are the superposition of a right and left traveling wave, with speed c.

* Search Dan Russell superposition and feel lucky.

We can see that this holds even for a ~~simple~~ Fourier standing wave,

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin(\sqrt{\lambda_n}\,x)\cos(c\sqrt{\lambda_n}\,t)$$

"Simple Plucked State" from Class

$$= \sum_{n=1}^{\infty} \frac{A_n}{2}\Big[\sin(\sqrt{\lambda_n}\,x - c\sqrt{\lambda_n}\,t) + \sin(\sqrt{\lambda_n}\,x + c\sqrt{\lambda_n}\,t)\Big]$$

$$= \underbrace{\sum_{n=1}^{\infty} \frac{A_n}{2}\sin(\sqrt{\lambda_n}(x-ct))}_{f(x-ct)} + \underbrace{\sum_{n=1}^{\infty} \frac{A_n}{2}\sin(\sqrt{\lambda_n}(x+ct))}_{g(x+ct)}$$

If we require that $u(x,0) = u_0(x)$

$$u_t(x,0) = v_0(x)$$

then

$$u(x,0) = f(x) + g(x) = u_0(x)$$

$$u_t(x,0) = -cf'(x) + c\,g'(x) = v_0(x)$$

$$\Rightarrow -c\left(u_0' - g'(x)\right) + c\,g'(x) = v_0(x)$$

$$\Rightarrow 2c\,g'(x) = v_0 + c\,u_0'$$

$$\Rightarrow g'(x) = \frac{v_0}{2c} + \frac{u_0'}{2}$$

$$\Rightarrow g(x) = \int_0^x \frac{v_0(s)}{2c}\,ds + \frac{u_0}{2}$$

$$\Rightarrow g(x+ct) = \int_0^{x+ct} \frac{v_0(s)}{2c}\,ds + \frac{u_0(x+ct)}{2}$$

$$\Rightarrow f(x-ct) = u_0(x-ct) - g(x-ct)$$

$$= u_0(x-ct) - \int_0^{x-ct} \frac{v_0(s)\,ds}{2c} - \frac{u_0(x-ct)}{2}$$

then

$$u(x,t) = f(x-ct) + g(x+ct) =$$

$$= \frac{u_0(x-ct)}{2} \underbrace{\underbrace{- \int_0^{x-ct} \frac{V_0(s)\,ds}{2c}}_{+\int_{x-ct}^{0}} + \int_0^{x+ct} \frac{V_0(s)}{2c}\,ds}_{} + \frac{u_0(x+ct)}{2}$$

$$= \frac{u_0(x-ct) + u_0(x+ct)}{2} + \int_{x-ct}^{x+ct} \frac{V_0(s)\,ds}{2c} \quad *$$

Is the representation of the superposition of right + left traveling waves that also obeys the initial conditions.

Notes:

- One could also check (*) by direct substitution and noting that

$$u(x,0) = \frac{u_0(x) + u_0(x)}{2} + \cancelto{0}{\int_x^x \text{stuff}} = u_0(x)$$

there is a lot of calc hidden in this term.

$$u_t(x,0) = \frac{-c\,u_0(x) + c\,u_0(x)}{2} + \frac{c V_0(x)}{2c} + \underbrace{\frac{c V_0(x)}{2c}}_{} = V_0(x)$$