Guided wave solutions

Wave equation:

Assume: no variation in structure with z

$n(x,y) = \sqrt{\varepsilon\mu}$ is piecewise constant, isotropic

then w/in each region,

$$\nabla \cdot \vec{D} = \nabla \cdot (\varepsilon \vec{E}) = \varepsilon \nabla \cdot \vec{E} = 0$$ no free charges

$\therefore$ use wave eqn. in each region

$$\nabla^2 \vec{E} - \frac{\varepsilon\mu}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$ where ε, μ depend on region

satisfy the boundary conditions to connect the solutions.

separation of variables

$$\vec{E}(\vec{r},t) = \vec{E}_T(\vec{r}_T)\, e^{i(k_z z - \omega t)}$$ wave prop to right if $k_z > 0$

All fields vary with $e^{-i\omega t}$.

Since eqn. is linear, we can later add different solutions with diff't ω's (e.g. Fourier series)

We can write Laplacian as

$$\nabla^2 \rightarrow \nabla_T^2 + \frac{\partial^2}{\partial z^2}$$

in Cartesian, $\nabla_T^2 f = (\partial_x^2 + \partial_y^2) f$

in cylindrical, $\nabla_T^2 f = \frac{1}{r}\partial_r (r \partial_r f) + \frac{1}{r^2}\partial_\phi^2 f$

Now wave eqn. is

$$\nabla_T^2 \vec{E}_T(\vec{r}_T) - k_z^2 \vec{E}_T + \frac{n^2\omega^2}{c^2}\vec{E}_T = 0$$

rearrange:

$$\nabla_T^2 \vec{E}_T + n^2(\vec{r}_T) k_0^2 \vec{E}_T = k_z^2 \vec{E}_T$$

w/ similar eqn for $\vec{B}_T$

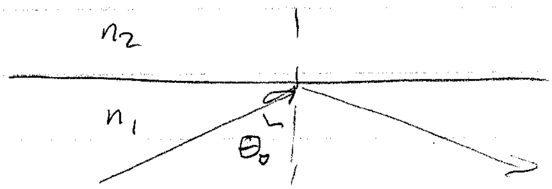
In Cartesian coord, allowed solutions are of form $e^{\pm i k_x x} e^{\pm i k_y y}$

$\rightarrow k_x^2 + k_y^2 + k_z^2 = n^2 k_0^2$ in each region

no assumptions here about k's and n's being real

Also, note that k_z is a constant through the system.

- each value of $n \rightarrow$ different values of $\vec{E}_T(\vec{r}_T)$; k_x, k_y
- all fields have $e^{i k_z z}$ dependence



$$k_z = n_1 k_0 \sin\theta_0 = n_2 k_0 \sin\theta_2$$

this is phase continuity = Snell's law

Structure of wave equation is an eigenvalue eqn:

$$\left[\nabla_T^2 + n^2(\vec{r}_T) k_0^2\right] \vec{E}_T = k_z^2 \vec{E}_T$$

same structure as Schrodinger wave eqn.

where $\hat{H}\Psi = E\Psi$

and $\hat{H} = \frac{\hat{p}^2}{2m} + V(\vec{r}) = -\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r})$

boundary conditions can impose quantization

$\rightarrow$ discrete eigenvalues k_z

revisit plane ∥ conductors; TE

$$\vec{E}_T = \hat{x}\left(a e^{ik_y y} + b e^{-ik_y y}\right)$$

$E = 0$ at $y = 0 \rightarrow a = -b$ $\therefore$ use $\sin k_y y$

$$\vec{E}_T = \hat{x} E_0 \sin(k_y y)$$

$E = 0$ at $y = b \rightarrow k_y b = m\pi$

$$k_y = \frac{m\pi}{b}$$

now $k_z = \sqrt{n^2 k_0^2 - k_y^2}$

TM:

$B_z = 0 \qquad E_y, E_z \neq 0 \qquad \nabla\cdot\vec{B} = \cancel{\partial_x B_x}^{\,0} + \partial_y B_y = 0$

$B_y = 0$ or constant, but at conductor $B_y \rightarrow 0$

$\therefore$ only have B_x, E_y, E_z

choose which to solve for.

redundancy in Maxwell eqns

$$\nabla\cdot\vec{E} = \partial_y E_y + \partial_z E_z = 0 \qquad ①$$

$$(\nabla\times\vec{E})_x = i k_0 B_x \rightarrow \underline{\partial_y E_z - i k_z E_y = i k_0 B_x} \qquad ②$$

$$\nabla\times B_x = -i k_0 \vec{E} \rightarrow \underline{i k_z B_x = -i k_0 E_y} \qquad ③$$

$$-\partial_y B_y = -i k_0 E_z \qquad ④$$

Any will work, but if we solve for E_z, can get others thru derivatives

③ $E_y = -\frac{k_z}{k_0} B_x$

$\rightarrow$ ④ $\partial_y E_z + i \frac{k_z^2}{k_0} B_x = i k_0 B_x$

$$k_0 \partial_y E_z = i(k_0^2 - k_z^2) B_x = i k_y^2 B_x$$

$$B_x = -i \frac{k_0}{k_y^2} \partial_y E_z$$

$$E_y = -\frac{k_z}{k_0} B_x = i \frac{k_z}{k_y^2} \partial_y E_z$$

message: can get all components for TM directly from E_z

for TE, can get all from B_z

This is especially valuable in 2D waveguides, since B.C. are uniform at all walls.

$\therefore$ for TM solve for E_z with $E_z \to 0$ at walls.

for TE solve for B_z

from $\nabla \times \vec{B} = -i k_0 \vec{E}$

$$\frac{\partial B_z}{\partial y} - i k_z B_y = i k_0 E_x$$

at walls $B_y = E_x = 0 \quad \therefore \quad \underline{\underline{\frac{\partial B_z}{\partial y} = 0}}$

<u>derivative of $B_z \to 0$ at walls</u> $\underline{\underline{TE}}$

HM Book has all connections btw fields derived.

just remember in 2D waveguides to solve for z compon