Homework #9 Solutions

1. Given,

$$ay'' + by' + cy = f(x) \qquad (1)$$

a) Find all solutions to the homogeneous version of (1).

$$\begin{aligned}
\text{Assume: } y &= e^{rx} & ar^2e^{rx} + bre^{rx} + ce^{rx} = 0\\
y' &= re^{rx} & ar^2 + br + c = 0\\
y'' &= r^2e^{rx} & r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{D}}{2a}, \text{ where } D = b^2 - 4ac
\end{aligned}$$

D	$y_c(x)$
> 0	$c_1 e^{(\frac{-b+\sqrt{D}}{2a})x} + c_2 e^{(\frac{-b-\sqrt{D}}{2a})x}$
$= 0$	$c_1 e^{(\frac{-b}{2a})x} + c_2 e^{(\frac{-b}{2a})x}$
< 0	$e^{(\frac{-b}{2a})x}[c_1 cos(\frac{\sqrt{D}}{2a}x) + c_2 sin(\frac{\sqrt{D}}{2a}x)$

b) Fill in the following table with the choices you would make for the particular solution.

$f(x)$	$y_c(x)$
x^4	$Ax^4 + Bx^3 + Cx^2 + Dx + E$
$cos(\beta x)$	$Acos(\beta x) + Bsin(\beta x)$
$e^{\alpha x} + sin(x) + x$	$Ae^{\alpha x} + Bcos(x) + Csin(x) + Dx + E$
$e^{\alpha x} sin(\beta x)$	$Ae^{\alpha x}cos(\beta x) + Be^{\alpha x}sin(\beta x)$
$x^2e^{\alpha x}cos(\beta x)$	$(Ax^2 + Bx + C)e^{\alpha x}cos(\beta x) + (Dx^2 + Ex + F)e^{\alpha x}sin(\beta x)$

c) Assume that a=1, b=0, c=9 and f(x)=cos(3x). Find the general solution to (1).
$D = b^2 - 4ac = -36$
From (a) we have that $y_c = c_1cos(3x) + c_2sin(3x)$

$$\begin{aligned}
y_p &= Axcos(3x) + Bxsin(3x)\\
y_p' &= -3Axsin(3x) + Acos(3x) + 3Bxcos(3x) + Bsin(3x)\\
y_p'' &= -9Axcos(3x) - 3Asin(3x) - 3Asin(3x) - 9Bxsin(3x) + 3Bcos(3x) + 3Bcos(3x)\\
&= -9Axcos(3x) - 6Asin(3x) - 9Bxsin(3x) + 6Bcos(3x)\\
y_p'' + 9y_p &= -9Axcos(3x) - 6Asin(3x) - 9Bsin(3x) + 6Bcos(3x) + 9Axcos(3x) + 9Bxsin(3x)\\
&= cos(3x)\\
&\Rightarrow -6Asin(3x) + 6Bcos(3x) = cos(3x) \Rightarrow \begin{matrix} A = 0 \\ B = \frac{1}{6} \end{matrix}\\
y_p &= \frac{1}{6}xsin(3x)\\
y_c + y_p &= y = c_1cos(3x) + c_2sin(3x) + \frac{x}{6}sin(3x)
\end{aligned}$$

2. Consider the boundary value problem

$$y'' + \lambda y = 0 \tag{2}$$
$$y'(0) = y'(1) = 0 \tag{3}$$

a) Assuming $\lambda \in R$, find the general solution to (2)

$$\begin{aligned} \text{Assume: } y &= e^{rx} \qquad & r^2e^{rx} + \lambda e^{rx} = 0 \\ y' &= re^{rx} \qquad & r^2 + \lambda = 0 \\ y'' &= r^2e^{rx} \qquad & r = \pm\sqrt{-\lambda} \end{aligned}$$

D	$y_c(x)$
> 0	$c_1e^{\sqrt{-\lambda}x} + c_2e^{-\sqrt{-\lambda}x} = C_1cosh(\sqrt{-\lambda}x) + C_2sinh(\sqrt{-\lambda}x)$
$= 0$	$c_1 + c_2x$
< 0	$c_1cos(\sqrt{\lambda}x) + c_2sin(\sqrt{\lambda}x)$

b) Apply the boundary conditions (3) and calculate the possible values of λ.

case 1: $\lambda < 0$

$$\begin{aligned} y'(x) &= [c_1sinh(\sqrt{-\lambda}x) + c_2cosh(\sqrt{-\lambda}x)]\sqrt{-\lambda} \\ y'(0) &= = 0 = c_2cosh(0) \Rightarrow c_2 = 0 \\ y'(1) &= c_1sinh(\sqrt{-\lambda}) \\ & \quad \text{because } \lambda \neq 0, c_1 = 0 \text{ and } y(x) = 0, \text{ the trivial solution} \end{aligned}$$

case 2: $\lambda = 0$

$$\begin{aligned} y'(x) &= c_2 \\ y'(0) &= = 0 = c_2 \\ y'(1) &= = 0 = c_2 \\ & \quad c_2 = 0, c_1 \in R \text{ Therefore } y(x) = c_1, \text{ a non-trivial solution} \\ & \quad \text{and 0 is a possible value of } \lambda. \end{aligned}$$

case 3: $\lambda > 0$

$$\begin{aligned} y'(x) &= -c_1\sqrt{\lambda}sin(\sqrt{\lambda}x) + c_2\sqrt{\lambda}cos(\sqrt{\lambda}x) \\ y'(0) &= = 0 = c_2\sqrt{\lambda} \Rightarrow \text{because } \lambda \neq 0, c_2 = 0 \\ y'(1) &= c_1\sqrt{\lambda}sin(\sqrt{\lambda}) \\ & \quad \text{assume } c_1 \neq 0, \lambda = n^2\pi^2, n = 1, 2, 3, \ldots, y(x) = c_1cos(\sqrt{\lambda}x) \end{aligned}$$

The possible values of λ are

$$\lambda = 0 \qquad \lambda = n^2\pi^2, n = 1, 2, 3, \ldots$$

And the general solution is

$$y(x) = c_0 + \sum_{n=1}^{\infty} c_ncos(n\pi)$$

3. Consider the ordinary differential equation:

$$y'' - y = 0 \tag{4}$$

a) Show that $y(x) = b_1 sinh(x) + b_2 cosh(x)$ is a solution to (4).

$$\begin{aligned}
y(x) &= b_1 sinh(x) + b_2 cosh(x) \\
y'(x) &= b_1 cosh(x) + b_2 sinh(x) \\
y''(x) &= b_1 sinh() + b_2 cosh(x) \\
y'' - y &= b_1 sinh(x) + b_2 cosh(x) - (b_1 sinh(x) + b_2 cosh(x)) = 0
\end{aligned}$$

b) Show that if $c_1 = \frac{b_1+b_2}{2}$ and $c_2 = \frac{b_1-b_2}{2}$ then

$$y(x) = c_1 e^x + c_2 e^{-x} = b_1 cosh(x) + b_2 sinh(x)$$

$$\begin{aligned}
y(x) &= c_1 e^x + c_2 e^{-x} \\
&= \left(\frac{b_1+b_2}{2}\right) e^x + \left(\frac{b_1-b_2}{2}\right) e^{-x} \\
&= \frac{b_1(e^x + e^{-x})}{2} + \frac{b_2(e^x - e^{-x})}{2} \\
&= b_1 cosh(x) + b_2 sinh(x)
\end{aligned}$$

The hyperbolic sine and cosine have the following taylor series representations:

$$cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \qquad sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \tag{5}$$

c) Assume that $y(x) = \sum_{n-0}^{\infty} a_n x^n$ and find the general solution of (4)

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_n x^n \qquad k = n \\
y'(x) &= \sum_{n=0}^{\infty} a_n (n) x^{n-1} \\
y''(x) &= \sum_{n=0}^{\infty} a_n (n)(n-1) x^{n-2} \qquad k = n-2 \\
y'' - y &= \, = 0 \\
&\Rightarrow \sum_{n=0}^{\infty} a_n (n)(n-1) x^{n-2} - \sum_{n=0}^{\infty} a_n x^n = 0 \\
& \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1) x^k - \sum_{k=0}^{\infty} a_k x^k = 0 \\
& \sum_{k=0}^{\infty} [a_{k+2}(k+2)(k+1) - a_k] x^k = 0
\end{aligned}$$

$$
\begin{aligned}
& && \text{since } x^k \text{ can't be 0, we assume} \\
& && a_{k+2}(k+2)(k+1) - a_k = 0 \\
& \Rightarrow && a_{k+2} = \frac{a_k}{(k+2)(k+1)}, k = 0, 1, 2, 3, \ldots \\
k = 0 & && a_2 = \frac{a_0}{2 \cdot 1} \\
k = 1 & && a_3 = \frac{a_1}{3 \cdot 2 \cdot 1} \\
k = 2 & && a_4 = \frac{a_2}{4 \cdot 3} = \frac{a_0}{4 \cdot 3 \cdot 2 \cdot 1} \\
k = 3 & && a_5 = \frac{a_3}{5 \cdot 4} = \frac{a_1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \\
k = 4 & && a_6 = \frac{a_4}{6 \cdot 5} = \frac{a_0}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \\
k = 5 & && a_7 = \frac{a_5}{7 \cdot 6} = \frac{a_1}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \\
& = && \text{for even k: } a_{2n} = \frac{a_0}{(2n)!}, \quad n = 0, 1, 2, 3, \ldots \\
& = && \text{for odd k: } a_{2n+1} = \frac{a_1}{(2n+1)!} \quad n = 0, 1, 2, 3, \ldots \\
y(x) & = && a_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \\
y(x) & = && a_0 cosh(x) + a_1 sinh(x)
\end{aligned}
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4. a) Explain the physical interpretation of the maximum principle.

Temperature comes from either a source or from earlier in time, but is not created from nothing.

b) Explain the relationship between the heat equation and the diffusion equation.

The diffusion equation is a non-linear form of the heat equation that is derived from the continuity equation using Fick's first law.

c) Write down the linear wave equation and explain c and c(u).

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

Where c is the propagation speed of the wave. c(u) is a wave speed dependent on the amplitude.

d) What physical phenomenon modeled by the linear wave equation? the non-linear wave equation?

Linear wave equation models waves and vibrations. The non-linear wave equation provides a more realistic model.

e) Define dispersion and give a physical example.

Dispersion is when the velocity of a wave is a function of frequency, such as sound waves traveling through water.

5. Show that the following functions are solutions to their corresponding PDE's.
a) u(x,t)=f(x-ct)+g(x+ct), 1-D wave equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= -df' + cg (\leftarrow \text{chain rule}) \qquad \frac{\partial u}{\partial x} = f' + g' \\ \frac{\partial^2 u}{\partial t^2} &= c^2 f'' + c^2 g'' \qquad \frac{\partial^2 u}{\partial x^2} = f'' + g'' \\ \text{1-D wave eqn} \rightarrow \frac{\partial^2 u}{\partial t^2} &= c^2 \frac{\partial^2 u}{\partial x^2} \Rightarrow c^2(f'' + g'') = c^2(f'' + g'') \end{aligned}$$

b) $u(x,t) = e^{-4w^2 t} sin(wx)$, 1-D heat eqn, c=2

$$\begin{aligned} \frac{\partial u}{\partial t} &= -4w^2 e^{-4w^2 t} sin(wx) \\ \frac{\partial u}{\partial x} &= we^{-4w^2 t} cos(wx) \qquad \frac{\partial^2 u}{\partial x^2} = -w^2 e^{-4w^2 t} sin(wx) \\ \text{1-D heat eqn, c=2} \rightarrow \frac{\partial u}{\partial E} &= (4) \frac{\partial^2 u}{\partial x^2} \\ \Rightarrow -4w^2 e^{-4w^2 t} sin(wx) &= -4w^2 e^{-4w^2 t} sin(wx) \end{aligned}$$

c) $u(x,t) = x^4 + y^4$, 2-D poisson eqn: $f(x,y) = 12(x^2 + y^2)$

$$\begin{aligned} \frac{\partial u}{\partial x} &= 4x^3 \qquad \frac{\partial u}{\partial y} = 4y^3 \\ \frac{\partial^2 u}{\partial x^2} &= 12x2 \qquad \frac{\partial^2 u}{\partial y^2} = 12y^2 \\ \text{2-D poisson eqn} \rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= f(x,y) = 12(x^2 + y^2) \\ \Rightarrow 12(x^2 + y^2) &= 12(x^2 + y^2) \end{aligned}$$

d) $u(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$, 3-D laplace equation

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{-x}{(x^2 + y^2 + z^2)^{3/2}} \\ \frac{\partial^2 u}{\partial x^2} &= \frac{3x^2}{(x^2 + y^2 + z^2)^{3/2}} + \frac{-1}{(x^2 + y^2 + z^2)^{3/2}} \\ \frac{\partial^2 u}{\partial y^2} &= \frac{3y^2}{(x^2 + y^2 + z^2)^{3/2}} + \frac{-1}{(x^2 + y^2 + z^2)^{3/2}} \\ \frac{\partial^2 u}{\partial z^2} &= \frac{3z^2}{(x^2 + y^2 + z^2)^{3/2}} + \frac{-1}{(x^2 + y^2 + z^2)^{3/2}} \end{aligned}$$

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\begin{aligned}
\text{3-D laplace eqn} \rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} &= 0 \\
&\Rightarrow \frac{3x^2 + 3y^2 + 3z^2}{(x^2 + y^2 + z^2)^{5/2}} + \frac{-3}{(x^2 + y^2 + z^2)^{3/2}} = 0 \\
&\Rightarrow \frac{3(x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^{5/2}} + \frac{-3}{(x^2 + y^2 + z^2)^{3/2}} = 0 \\
&\Rightarrow \frac{3}{(x^2 + y^2 + z^2)^{3/2}} + \frac{-3}{(x^2 + y^2 + z^2)^{3/2}} = 0 \\
\Rightarrow 0 &= 0
\end{aligned}
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