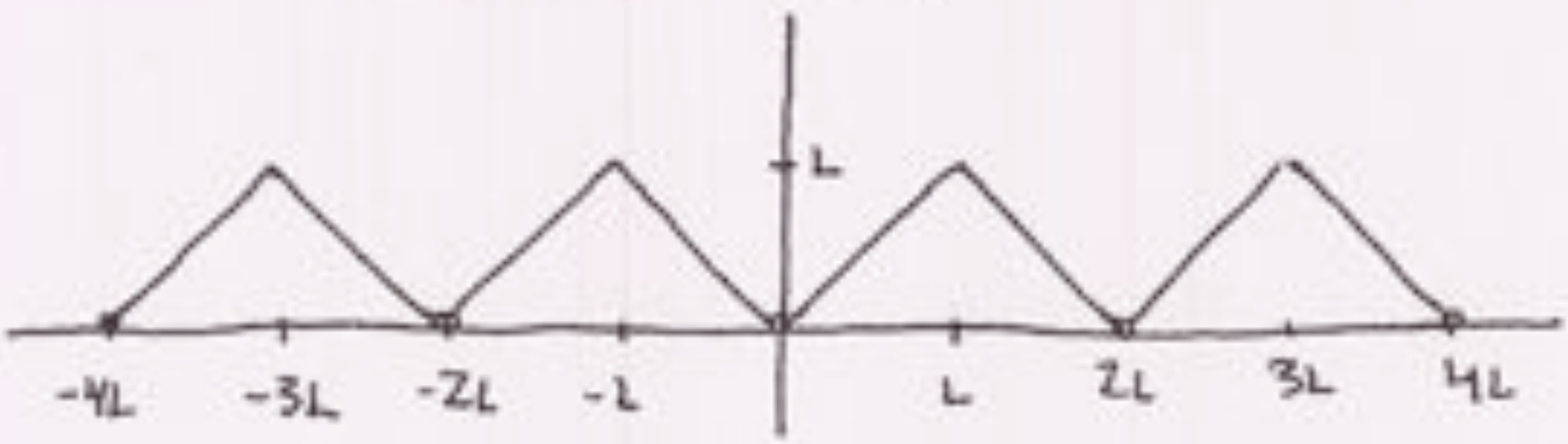
MATH 348 - SPRING 2008

HOMEWORK 3

[1] GIVEN
$$f(x)=\begin{cases} x & 0<x\le L \\ -x+2L & L<x<2L \end{cases}$$

(a) SKETCH THE GRAPH OF $f(x)$ ON $[-4L, 4L]$

(b) SKETCH f^*, THE EVEN PERIODICALLY EXTENDED VERSION OF f ON $[-4L, 4L]$



(c) CALCULATE THE FOURIER COSINE SERIES

$$f(x)=\begin{cases} -x & -L<x<0 \\ x & 0<x<L \end{cases}$$

FORMULAS FROM KREYZSIG P.491

$$f(x)=a_0+\sum_{n=1}^{\infty} a_n\cos\left(\tfrac{n\pi}{L}x\right)$$

$$a_0=\frac{1}{L}\int_0^L f(x)\,dx=\frac{1}{L}\int_0^L x\,dx=\frac{1}{2L}x^2\Big|_0^L=\frac{L}{2}$$

$$a_n=\frac{2}{L}\int_0^L f(x)\cos\left(\tfrac{n\pi}{L}x\right)dx$$

$$=\frac{2}{L}\int_0^L x\cos\left(\tfrac{n\pi}{L}x\right)dx$$

$$=\frac{2}{L}\left[\frac{xL}{n\pi}\sin\left(\tfrac{n\pi}{L}x\right)+\frac{L^2}{n^2\pi^2}\cos\left(\tfrac{n\pi}{L}x\right)\right]_0^L$$

MATH 348 - SPRING 2008

HOMEWORK 3

$$= \frac{2}{L}\left[\frac{L^2}{n\pi}\sin(n\pi) + \frac{L^2}{n^2\pi^2}\cos(n\pi) - 0 - \frac{L^2}{n^2\pi^2}\cos(0)\right]$$

$$= \frac{2}{L}\left[\frac{L^2}{n^2\pi^2}\left[(-1)^n - 1\right]\right]$$

$$= \frac{2L[(-1)^n - 1]}{n^2\pi^2}$$

$$\boxed{f(x) = \frac{L}{2} + \sum_{n=1}^{\infty} \frac{2L[(-1)^n - 1]}{n^2\pi^2}\cos\left(\frac{n\pi}{L}x\right)}$$

$\boxed{2}$

(a) SHOW THAT $\int_{-\pi}^{\pi} e^{inx} e^{-imx}\,dx = 2\pi\delta_{mn}$ WHERE $m, n \in \mathbb{R}$

FOR $n \neq m$: $\int_{-\pi}^{\pi} e^{inx}e^{-imx}\,dx = \int_{-\pi}^{\pi} e^{(n-m)ix}\,dx = \frac{e^{(n-m)ix}}{i(n-m)}\Big|_{-\pi}^{\pi}$

$$= \frac{(-1)^{(n-m)}}{i(n-m)} - \frac{(-1)^{(n-m)}}{i(n-m)} = 0$$

FOR $n = m$: $\int_{-\pi}^{\pi} e^{inx}e^{-imx}\,dx = \int_{-\pi}^{\pi} e^{(n-m)ix}\,dx = \int_{-\pi}^{\pi} 1\,dx = x\Big|_{-\pi}^{\pi} = 2\pi$

$$\int_{-\pi}^{\pi} e^{inx}e^{-imx}\,dx = \begin{cases} 0 & n \neq m \\ 2\pi & n = m \end{cases} = 2\pi\delta_{nm}$$

(b) FIND THE FOURIER COEFFICIENTS OF $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$

$$f(x) = \sum_{n=\infty}^{\infty} c_n e^{inx}$$

$$\Rightarrow f(x)e^{-imx} = \sum_{n=-\infty}^{\infty} c_n e^{inx}e^{-imx}$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x)e^{-imx}\,dx = \sum_{n=-\infty}^{\infty}\int_{-\pi}^{\pi} c_n e^{(n-m)x}\,dx$$

AS WE FOUND IN (a), THE INTEGRAL ON THE RIGHT IS 0 FOR ALL VALUES OF n EXCEPT FOR $n = m$

MATH 348 - SPRING 2008

HOMEWORK 3

$$\Rightarrow \int_{-\pi}^{\pi} f(x) e^{-imx} dx = c_m 2\pi$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-imx} dx = c_m$$

$$\boxed{c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} dx}$$

BECAUSE $m=n$ WE CAN REPLACE OUR m'S WITH n'S TO GET THE FORMULA FOR c_n

$\boxed{3}$ LET $f(x)=x^2$ $-\pi<x<\pi$ BE 2π-PERIODIC

(a) CALCULATE THE COMPLEX FOURIER SERIES REPRESENTATION OF $f(x)$

FORMULAS FROM KREYSZIG P. 497

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 e^{-inx} dx$$

$$= \frac{1}{2\pi}\left[\frac{-x^2}{in} e^{-inx} + \frac{2x}{n^2} e^{-inx} + \frac{2}{in^3} e^{-inx}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[\left(\frac{-x^2}{in} + \frac{2x}{n^2} + \frac{2}{in^3}\right) e^{-inx}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[\left(\frac{-\pi^2}{in} + \frac{2\pi}{n^2} + \frac{2}{in^3} + \frac{\pi^2}{in} + \frac{2\pi}{n^2} - \frac{2}{in^3}\right)(-1)^n\right]$$

$$= \frac{1}{2\pi}\left[\frac{4\pi}{n^2}(-1)^n\right] = \frac{2\pi}{n^2}(-1)^n \quad n \neq 0$$

FOR $n=0$

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 e^{i(0)x} dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{1}{2\pi}\left[\frac{1}{3}x^3\right]_{-\pi}^{\pi}$$

$$= \frac{\pi^2}{3}$$

$$\boxed{f(x) = \frac{\pi^2}{3} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{2\pi}{n^2}(-1)^n e^{inx}}$$

(b) USING (a), RECOVER THE REAL FOURIER SERIES REPRESENTATION OF $f(x)$.

$$f(x) = \frac{\pi^2}{3} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{2\pi}{n^2}(-1)^n e^{inx}$$

$$= \frac{\pi^2}{3} + \sum_{n=-\infty}^{-1} \frac{2\pi}{n^2}(-1)^n e^{inx} + \sum_{n=1}^{\infty} \frac{2\pi}{n^2}(-1)^n e^{inx}$$

SUBSTITUTING $n=-n$ INTO THE FIRST SERIES WE GET:

$$= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{2\pi}{n^2}(-1)^n e^{-inx} + \sum_{n=1}^{\infty} \frac{2\pi}{n^2}(-1)^n e^{inx}$$

$$= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{2\pi}{n^2}(-1)^n \left(e^{-inx} + e^{inx}\right)$$

USING EULER'S FORMULA:

$$= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{2\pi}{n^2}(-1)^n \left(\cos(nx) - i\sin(nx) + \cos(nx) + \sin(nx)\right)$$

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4\pi}{n^2}(-1)^n \cos(nx)$$

THE REAL FOURIER SERIES REPRESENTATION OF $f(x)$

1. How is the Fourier transform related to Fourier series? You should discuss both the periodicity and number of Fourier modes used in the construction of each.

   FT and FS are the same in that they take a function and represent it as the sum of oscillatory functions multiplied by amplitudes of oscillations. However, they differ by the number of Fourier modes (terms in the summations) used. In the case of FS there are countably infinite number of oscillatory functions which depend on a countably infinite frequency spectrum and these modes are used to construct periodic functions. In the case of FT the spectrum must be continuous and thus there are uncountably infinite modes depending on a continuum of frequencies.

2. What does cross-correlation measure? What would auto-correlation measure?

   Correlation is a measure of similarity between two functions. This measure is given in terms of a convolution integral. If cross-correlation is the measure of similarity between two functions then auto-correlation is a measure between the function and itself.

3. What is the uncertainty principle as it relates to Fourier transforms? How much power would be required to send a signal like $\delta(t)$?

   The uncertainty principle for FT says that if a function is localized in one domain then it is de-localized in the transformed domain. One can interpete this physically as saying if you know very well the spread of the function in one space then you know very little about the spread in another space. These two spaces are typically position-momentum or time-frequency. Using this idea one can show that the FT of a delta function is a constant function and the power of the signal is the area under the square of this constant function, which is infinite. Therefore you would need an infinite amount of power to send a single impulse of signal.

MATH 348 - SPRING 2008

HOMEWORK 4

$\boxed{1}$ $$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(w)e^{iwx}dw \quad (1)$$

$$\hat{f}(w)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-iwx}dx \quad (2)$$

$$f_c(x)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}\hat{f}(w)\cos(wx)dw \qquad \hat{f}_c(w)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}f(x)\cos(wx)dx \quad (3)$$

$$f_s(x)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}\hat{f}(w)\sin(wx)dw \qquad \hat{f}_s(w)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}f(x)\sin(wx)dx \quad (4)$$

(a) SHOW THAT $f_c(x)$ AND $\hat{f}_c(w)$ ARE EVEN FUNCTIONS AND $f_s(x)$ AND $\hat{f}_s(w)$ ARE ODD FUNCTIONS.

IN $f_c(x)$, $\cos(wx)$ IS THE ONLY FUNCTION OF X

SO THAT $f_c(-x)=f_c(x)$ THE DEFINITION OF AN EVEN FXN.

SIMILARLY $\hat{f}_c(-w)=\hat{f}_c(w)$

IN $f_s(x)$, $\sin(wx)$ IS THE ONLY FUNCTION OF X

SO THAT $f_s(-x)=-f_s(x)$ THE DEFINITION OF AN ODD FXN.

SIMILARLY $\hat{f}_s(-w)=-\hat{f}_s(w)$

(b) SHOW THAT, IF WE ASSUME $f(x)$ IS EVEN, (1)-(2) DEFINES THE TRANSFORM PAIR (3) AND IF WE ASSUME $f(x)$ IS ODD, (1)-(2) DEFINES THE TRANSFORM PAIR (4)

USING EULER'S FORMULA, (2) BECOMES

$$\hat{f}(w)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)(\cos(wx)-i\sin(wx))dx$$

$$\hat{f}(w)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)\cos(wx)dx-\frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)\sin(wx)dx$$

IF WE ASSUME $f(x)$ IS EVEN, IS AN ODD FUNCTION INTEGRATED OVER SYMMETRIC BOUNDS, WHICH BECOMES 0

AND $\hat{f}(w)=\hat{f}_c(w)$

BECAUSE $\hat{f}(w) = \hat{f}_c(w)$, WHICH IS EVEN, [redacted] THE SAME ARGUMENT CAN BE USED TO SHOW THAT $f(x) = f_c(x)$

IF WE ASSUME $f(x)$ IS ODD

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx}\, dx$$

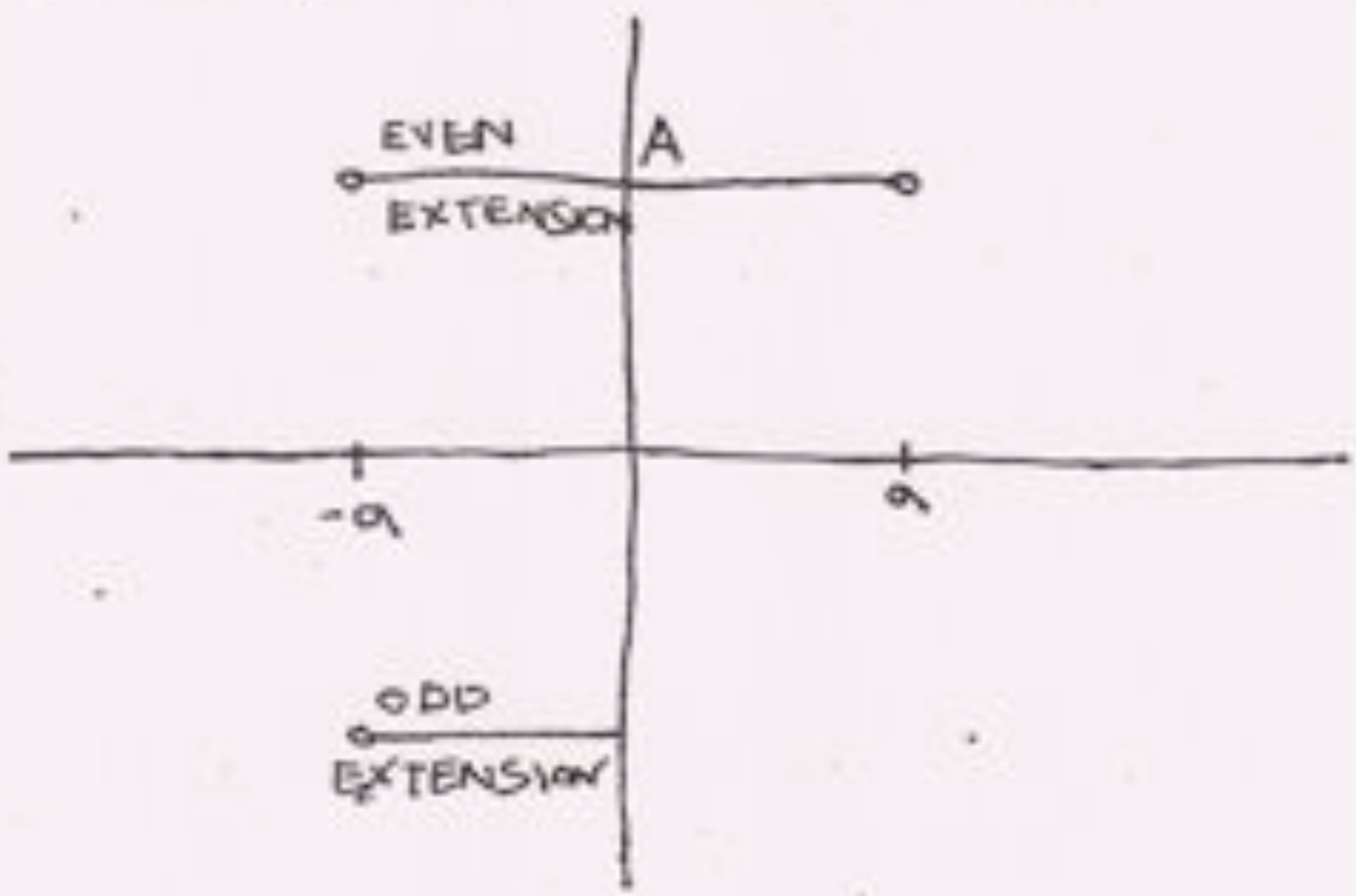
$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\cos(wx)\, dx - \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\sin(wx)\, dx$$

BECAUSE $f(x)$ IS ODD, ↑ BECOMES AN ODD FUNCTION INTEGRATED OVER SYMMETRIC BOUNDS WHICH BECOMES 0 AND $\hat{f}(w) = \hat{f}_s(w)$

BECAUSE $\hat{f}(w) = \hat{f}_s(w)$, WHICH IS ODD, THE SAME ARGUMENT CAN BE USED TO SHOW THAT $f(x) = f_s(x)$

✓ (c) $f(x) = \begin{cases} A & 0 < x < a \\ 0 & \text{OTHERWISE} \end{cases}$ $A, a \in \mathbb{R}^+$ (5)

PLOT THE EVEN AND ODD EXTENSIONS OF $f(x)$.



MATH 348 - SPRING 2008

HOMEWORK 4

(d) FIND THE FOURIER COSINE AND SINE TRANSFORMS OF $f(x)$.

$$\hat{f}_c(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\cos(wx)\,dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^a A\cos(wx)\,dx = \sqrt{\frac{2}{\pi}} \left[\frac{A}{w}\sin(wx)\right]_0^a$$

$$\boxed{\hat{f}_c(w) = \sqrt{\frac{2}{\pi}}\left(\frac{A\sin(aw)}{w}\right)}$$

$$\hat{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\sin(wx)\,dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^a A\sin(wx)\,dx = \sqrt{\frac{2}{\pi}} \left[\frac{-A}{w}\cos(wx)\right]_0^a.$$

$$\boxed{\hat{f}_s(w) = -\sqrt{\frac{2}{\pi}}\left(\frac{A(\cos(aw)-1)}{w}\right)}$$

(e) USING THE FOURIER COSINE TRANSFORM, SHOW THAT $\int_{-\infty}^{\infty} \frac{\sin(w\pi)}{w\pi}\,dw = 1$

FROM (d) WE HAVE THAT IF $f(x) = \begin{cases} A & 0<x<a \\ 0 & \text{OTHERWISE} \end{cases}$

$$\hat{f}_c(w) = \sqrt{\frac{2}{\pi}}\left(\frac{A\sin(wa)}{w}\right)$$

IF WE TAKE THE INVERSE TRANSFORM

$$\mathcal{F}^{-1}\{\hat{f}_c(w)\} = f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \left[\sqrt{\frac{2}{\pi}}\left(\frac{A\sin(aw)}{w}\right)\right]\cos(wx)\,dw$$

EVALUATING $f(x)$ AT 0

$$f(0) = \frac{2}{\pi}\int_0^\infty \frac{A\sin(wa)}{w}\,dw = A$$

MATH 348 - SPRING 2008

HOMEWORK 4

BECAUSE $\frac{A\sin(wa)}{w}$ IS AN EVEN FUNCTION,

$$\frac{2}{\pi}\int_0^\infty \frac{A\sin(wa)}{w}\,dw = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{A\sin(wa)}{w}\,dw = A$$

IF WE CHOOSE $A=1$ AND $a=\pi$,

$$\int_{-\infty}^{\infty}\frac{\sin(w\pi)}{w\pi}\,dw = 1$$

[2] CALCULATE THE FOLLOWING TRANSFORMS

(a) $\mathfrak{F}\{f\}$ WHERE $f(x)=\delta(x-x_0)$, $x_0\in\mathbb{R}$

$$\hat{f}(w)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(x-x_0)e^{-iwx}\,dx$$

$$=\boxed{\frac{1}{\sqrt{2\pi}}e^{-iwx_0}}$$

(b) $\mathfrak{F}\{f\}$ WHERE $f(x)=e^{-k_0|x|}$, $k_0\in\mathbb{R}^+$

$$f(x)=\begin{cases} e^{-k_0x} & x>0 \\ e^{k_0x} & x<0\end{cases}$$

$$\hat{f}(w)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}e^{k_0x}e^{iwx}\,dx+\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-k_0x}e^{iwx}\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\left[\frac{e^{(k_0+iw)x}}{k_0+iw}\Big|_{-\infty}^{0}+\frac{e^{(iw-k_0)x}}{iw-k_0}\Big|_0^\infty\right]$$

$$=\frac{1}{\sqrt{2\pi}}\left[\frac{1}{k_0+iw}-\frac{1}{iw-k_0}\right]=\frac{1}{\sqrt{2\pi}}\left[\frac{-2k_0}{-w^2-k_0^2}\right]$$

$$\boxed{=\sqrt{\frac{2}{\pi}}\left(\frac{k_0}{w^2+k_0^2}\right)}$$