Potentials : Gauge Transforms.

Classical Radiation: Dipole Radiation.

Special Relativity: Cool.

Today: Potential Griffith's 10.1.

Tomorrow: Griffiths 12.1, 12.2.

Potentials: What are they?

Voltage,

Back in the day, you guys used voltage, V, and the magnetic potential, $\vec{A}$, in electrostatics.

$$\vec{E} = -\vec{\nabla} V \quad ; \quad \vec{B} = \vec{\nabla}\times\vec{A}$$

Why? Why?

$\vec{\nabla}\times\vec{E} = \emptyset$ $\qquad$ $\vec{\nabla}\cdot\vec{B} = \emptyset$

$\vec{\nabla}\times(\vec{\nabla} V) = \emptyset$ in general. $\qquad$ $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = \emptyset$ in general.

was true for electro statics, but it's not for electrodynamics

$$\underbrace{\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}} \Rightarrow \vec{B} \neq \text{const. in time} \Rightarrow \vec{E} \neq -\vec{\nabla} V$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{A})$$

$$= -\vec{\nabla}\times\left(\frac{\partial\vec{A}}{\partial t}\right)$$

$$\Rightarrow \vec{\nabla}\times\vec{E} + \vec{\nabla}\times\left(\frac{\partial\vec{A}}{\partial t}\right) = \emptyset$$

$$\vec{\nabla}\times\left(\underbrace{\vec{E} + \frac{\partial\vec{A}}{\partial t}}\right) = \emptyset$$

$$\nabla\times(\text{this thing}) = \emptyset$$

$$\Rightarrow \vec{E} + \frac{\partial\vec{A}}{\partial t} = -\vec{\nabla} V$$

$$\boxed{\vec{E} = -\vec{\nabla} V - \frac{\partial\vec{A}}{\partial t} \quad ; \quad \vec{B} = \vec{\nabla}\times\vec{A}}$$

$\Rightarrow$ Show $\vec{\nabla}\cdot\vec{B} = 0$ & $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ are automatically satisfied using the above eqns.

$\Rightarrow$ Plug it in to the other two Maxwell's eqns

The other maxwell equs give

$$\nabla^2 V + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = \frac{-\rho}{\epsilon_0}$$

$$\nabla^2\vec{A} - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2} - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t}\right) = -\mu_0\vec{J}$$

These two equs with $\vec{B} = \vec{\nabla}\times\vec{A}$; $\vec{E} = -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t}$ are equivalent to Maxwell's equs... sort of.

This looks crappy, but we can play around with it a little. The reason is all we care about is that $\vec{\nabla}\times\vec{A} = \vec{B}$ and $-\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t} = \vec{E}$. $\Rightarrow$ we can change $\vec{A}, V$ just as long as $\vec{E}$ and $\vec{B}$ aren't effected.

$$\vec{A}' = \vec{A} + \vec{\alpha} \quad ; \quad V' = V + \beta$$

we need $\vec{\nabla}\times\vec{A}' = \vec{\nabla}\times\vec{A}$ & $-\vec{\nabla}V' - \frac{\partial\vec{A}'}{\partial t} = -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t}$

$$\vec{\nabla}\times(\vec{A}+\vec{\alpha}) = \vec{\nabla}\times\vec{A}$$

$$\Rightarrow \vec{\nabla}\times\vec{\alpha} = \emptyset$$

Let's rewrite $\vec{\alpha}$ as $\vec{\nabla}\lambda$

$$\Rightarrow \vec{\nabla}\times(\vec{\nabla}\lambda) = \emptyset \checkmark$$

$$\Rightarrow -\vec{\nabla}(V+\beta) - \frac{\partial}{\partial t}(\vec{A}+\vec{\nabla}\lambda) =$$

$$\Rightarrow -\vec{\nabla}\beta - \frac{\partial}{\partial t}\vec{\nabla}\lambda = \emptyset$$

$$\vec{\nabla}\left(\beta + \frac{\partial\lambda}{\partial t}\right) = \emptyset$$

$$\beta = -\frac{\partial\lambda}{\partial t} + \underline{\text{Const.}}$$

Gist is we can transform $V, \vec{A}$ using the following transformation

$$V' = V - \frac{\partial\lambda}{\partial t} \quad ; \quad \vec{A}' = \vec{A} + \vec{\nabla}\lambda$$

for any $\lambda(\vec{r}, t)$. This is what a gauge transformation is.

Two common gauges!

Coloumb Gauge: we pick λ so $\vec{\nabla}\cdot\vec{A} = \emptyset$.

$\Rightarrow$ Maxwell's eqns give:

$$\nabla^2 V = \frac{-\rho}{\epsilon_0} \quad ; \quad \nabla^2\vec{A} - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J} + \mu_0\epsilon_0\vec{\nabla}\left(\frac{\partial V}{\partial t}\right)$$

finding V is easy, finding $\vec{A}$ is hard.

Lorentz Gauge: we pick λ so $\vec{\nabla}\cdot\vec{A} = -\mu_0\epsilon_0\frac{\partial V}{\partial t}$

$\Rightarrow$ Plug this into Maxwell's eqn's and see what happens.

$$\left(\nabla^2 - \mu_0\epsilon_0\frac{\partial^2}{\partial t^2}\right)\vec{A} = -\mu_0\vec{J}$$

$$\left(\nabla^2 - \mu_0\epsilon_0\frac{\partial^2}{\partial t^2}\right)V = \frac{-\rho}{\epsilon_0}$$

Some interesting things from Chapter 10

$$V(\vec{r},t) = \int_{\text{all space}} \frac{\rho(\vec{r}\,',t_r)}{\mathscr{r}}\, d\tau' \qquad \vec{A} = \int_{\text{all space}} \frac{\vec{J}(\vec{r}\,',t_r)}{\mathscr{r}}\, d\tau'$$

$$\mathscr{r} = |\vec{r} - \vec{r}\,'| \qquad t_r = t - \frac{\mathscr{r}}{c}.$$

The related expressions for $\vec{E}, \vec{B}$.

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int_{\text{all space}} \left[\frac{\rho(\vec{r}\,',t_r)}{\mathscr{r}^2}\hat{\mathscr{r}} + \frac{\dot{\rho}(\vec{r}\,',t_r)}{c\mathscr{r}}\hat{\mathscr{r}} - \frac{\dot{\vec{J}}(\vec{r}\,',t_r)}{c^2\mathscr{r}}\right] d\tau'$$

$$\vec{B} = \frac{\mu_0}{4\pi}\int_{\text{all space.}} \left[\frac{\vec{J}(\vec{r}\,',t_r)}{\mathscr{r}^2} + \frac{\dot{\vec{J}}(\vec{r}\,',t_r)}{c\mathscr{r}}\right] d\tau'$$