Acoustics and the Wave Equation

What do we hear and how do we hear it?

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Overview/Keywords/References

Advanced Engineering Mathematics **Slide Set Seven**

Polar Geometries and Small Amplitude Vibrations

Reference Text: EK 12.9

- See Also:
  - Lecture Notes : 15.LN.WaveEquation

Before We Begin

Quote of Slide Set Seven
All these squawking birds won't quit. Building nothing, laying bricks.
The Shins : Caring Is Creepy (2001)

Acoustics

Acoustics is the study of sound and **sound is a traveling wave** which is an oscillation of pressure transmitted through a solid, liquid, or gas, composed of frequencies within the range of hearing and of a level sufficiently strong to be heard, or the sensation stimulated in organs of hearing by such vibrations.

- The model equation for the evolution of this traveling wave is unclear.

One would hope that the linear wave equation is appropriate but hope is far from rigor.
Problem: Using continuum mechanics, derive an equation modeling traveling waves in a pressure field.

Conservation Equations - Part I

Recall that in our derivation of the heat/diffusion equation we obtained the following conservation principle,

$$u_t + \text{div}(\phi) = f(\mathbf{x}, t), \quad \mathbf{x} \in \mathbb{R}^3, \tag{1}$$

where u is a density (stuff/unit volume) and ϕ is its flux (stuff/unit area $\cdot$ unit time). For acoustics we adopt the following,

- Mass Density : $u = \rho$
- Mass Flux : $\phi = \rho\boldsymbol{\nu}$, where $\boldsymbol{\nu} \in \mathbb{R}^3$ is the velocity field.

Conservation of mass implies,

$$\rho_t + \text{div}(\rho\boldsymbol{\nu}) = 0. \tag{2}$$

Conservation Equations - Part II

Momentum must also be conserved. In this case we have,

- Momentum Density : $u = \rho\boldsymbol{\nu}$
- Momentum Flux : $\phi = \rho\boldsymbol{\nu}^2$ where $\left[\boldsymbol{\nu}^2\right]_i = \nu_i^2$
- Newton's Second Law : $f(\mathbf{x}, t) = -\nabla p$ where $p(\mathbf{x}, t)$ is the pressure field

Conservation of momentum now reads,

$$(\rho[\boldsymbol{\nu}]_i)_t + \text{div}(\rho\boldsymbol{\nu}^2) = -[\nabla p]_i, \quad i = 1, 2, 3. \tag{3}$$

Problem: This system is nonlinear and has fewer equations (four) than unknowns (five) and thus must be closed by some constitutive relation.

Constitutive Relation

It is sensible to assume that pressure is a function of mass density. It is typical to assume that,

$$p = F(\rho) = \kappa \rho^{\gamma}, \quad \kappa > 0, \quad \gamma > 1 \tag{4}$$

so that, $F'(\rho) > 0$, pressure is an increasing function of density. However, things simplify under the assumption that the pressure-field undergoes SMALL DISTURBANCES. That is,

$$p = F(\rho_0 + \tilde{\rho}) = F(\rho_0) + F'(\rho_0)\tilde{\rho} + \cdots = p_0 + c^2 \tilde{\rho} + \cdots \tag{5}$$

where $\rho = \rho_0 + \tilde{\rho}$ represents a small perturbation, $\tilde{\rho}$ from the rest pressure-field ρ_0. Note that,

$$[c] = \left[\sqrt{F'(\rho_0)}\right] = \frac{\text{length}}{\text{time}}, \tag{6}$$

is called the sound-speed of the acoustic medium.

Linearization

Now that the equations close we rewrite them neglecting products of small terms:

$$\rho_t + \text{div}(\rho\boldsymbol{\nu}) = (\rho_0 + \tilde{\rho})_t + \text{div}((\rho_0 + \tilde{\rho})\tilde{\boldsymbol{\nu}}) \tag{7}$$

$$= \tilde{\rho}_t + \rho_0 \text{div}(\tilde{\boldsymbol{\nu}}) = 0, \tag{8}$$

$$(\rho[\boldsymbol{\nu}]_i)_t + \text{div}(\rho\boldsymbol{\nu}^2) + [\nabla p]_i = ((\rho_0 + \tilde{\rho})[\tilde{\boldsymbol{\nu}}]_i)_t + \text{div}((\rho_0 + \tilde{\rho})\tilde{\boldsymbol{\nu}}^2) + \tag{9}$$

$$+ \left[\nabla \left(p_0 + c^2\tilde{\rho} + \cdots\right)\right]_i \tag{10}$$

$$= \rho_0\tilde{\boldsymbol{\nu}}_t + c^2\nabla\tilde{\rho} = 0. \tag{11}$$

These equations are called the ACOUSTIC APPROXIMATION equations and model the evolution of small deviations $\tilde{\rho}$ and $\tilde{\boldsymbol{\nu}}$ from the ambient state ρ_0 and $\boldsymbol{\nu} = \mathbf{0}$.

Reduction to Linear Wave Equations

It is possible to reduce (8) and (11) to wave equations. First, take the time-derivative of (8) and the divergence of (11) to get,

$$\tilde{\rho}_{tt} + \rho_0 \text{div}(\tilde{\boldsymbol{\nu}})_t = \tilde{\rho}_{tt} + \rho_0 \text{div}(\tilde{\boldsymbol{\nu}}_t) \tag{12}$$

$$= \tilde{\rho}_{tt} - c^2 \text{div}\left(\nabla \tilde{\rho}\right) \tag{13}$$

$$= \tilde{\rho}_{tt} - c^2 \triangle \tilde{\rho} = 0. \tag{14}$$

Similarly, for an irrotational velocity field, if we take the gradient of (8) and the time-derivative of (11) we get,

$$\rho_0 \tilde{\boldsymbol{\nu}}_{tt} + c^2 \nabla \tilde{\rho}_t = \rho_0 \tilde{\boldsymbol{\nu}}_{tt} - \rho_0 c^2 \nabla \text{div}(\tilde{\boldsymbol{\nu}}) \tag{15}$$

$$= \rho_0 \tilde{\boldsymbol{\nu}}_{tt} - \rho_0 c^2 \triangle \tilde{\boldsymbol{\nu}} = 0, \tag{16}$$

which implies that small disturbances are propagated through an acoustic medium by linear wave equations.

Conclusions - Part I

At this point we the following conclusions:

1. The conservation equations for density and velocity in an acoustic medium are nonlinear and underdetermined.
2. If we consider the pressure to be a function of the density then we can close the equations.
3. If we consider a Taylor expansion of this pressure function truncated to linear terms then the conservation equations linearize.
4. In this acoustic approximation small disturbances to the medium are propagated via linear wave equations.
5. In the case of large disturbances the nonlinearity cannot be neglected. The nonlinear equations predict the occurrence of so-called shock-waves.

Vibrations of an Ideally Elastic Circle

So, at this point we can consider the small amplitude vibrations of an ideally elastic circular membrane, model the dynamics human eardrum occurring right now. The PDE reads,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \triangle u, \tag{17}$$

where the physical domain is $D = \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq R^2\right\}$ for some $R \in \mathbb{R}^+$. For something like a drumhead we require,

$$u(x, y) = 0, \quad \text{on } \partial D. \tag{18}$$

Key Point: The coordinate system should always be chosen so that the boundary condition is as easy to express as possible.

$\triangle u$ in Polar Coordinates

To make the boundary condition easy to express we choose polar coordinates, $u(x, y, t) \to u(r, \theta, t)$. However, this yields a new problem: $u_x(r, \theta) = ?$. This is a question for the multivariate chain rule.

$$
\begin{aligned}
u_x(r, \theta) = u_r r_x + u_\theta \theta_x \Rightarrow \\
&\Rightarrow u_{xx} = (u_r r_x) + (u_\theta \theta_x) = u_{rx} r_x + u_r r_{xx} + u_{\theta x} \theta_x + u_\theta \theta_{xx} \\
&= u_{rr} r_x^2 + u_r r_{xx} + u_{\theta\theta} \theta_x^2 + u_\theta \theta_{xx}.
\end{aligned}
$$

A similar result holds for y. Computing these derivatives and simplifying gives,

$$\triangle u = u_{xx} + u_{yy} = u_{rr} + r^{-1} u_r + r^{-2} u_{\theta\theta} \tag{19}$$

Full PDE

The Laplacian to polar coordinates gives the full PDE as:

$$u_{tt} = c^2 \left(u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta}\right) \tag{20}$$

$$(r,\theta) \in (0,R) \times (-\pi,\pi] \tag{21}$$

$$u(r,\theta,0) = f(r,\theta) \tag{22}$$

$$u_t(r,\theta,0) = g(r,\theta) \tag{23}$$

$$u(R,\theta,t) = 0 \tag{24}$$

We will see that this gives a very different solution than the similar problem in Cartesian coordinates whose solution is,

$$u(x,y,t) =$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} [B_{nm}\cos(c\lambda_{nm} + B^*_{nm}\sin(c\lambda_{nm}t)] \sin\left(\frac{m\pi}{L_1}x\right)\sin\left(\frac{n\pi}{L_2}y\right)$$

Separation of Variables

Assume that $u(r, \theta, t) = F(r, \theta)G(t)$ to get,

$$\ddot{G} + c^2\lambda G = 0, \tag{25}$$

$$F_{rr} + r^{-1}F_r + r^{-2}F_{\theta\theta} + \lambda F = 0 \tag{26}$$

At this point one could assume that $F(r, \theta) = W(r)Q(\theta)$ and separate again. However, it should be clear that $Q(\theta + 2\pi) = Q(\theta) = \sin(n\theta), \cos(n\theta)$. Thus, the spatial equation now reads,

$$W'' + r^{-1}W' - n^2r^{-2}W + \lambda W = 0, \tag{27}$$

or

$$r^2W'' + rW' + (r^2\lambda - n)W = 0. \tag{28}$$

Bessel's Equation - Part I

We now consider the transformation $s = \sqrt{\lambda} r$ on the previous equation to get,

$$s^2 W'' + sW' + (s - n)W = 0. \tag{29}$$

where the derivative now indicates a derivative with respect to the s variable. This is nothing more than Bessel's equation of order n and we know the solutions to be,

$$J_n(s) = s^n \sum_{m=0}^{\infty} \frac{(-1)^m s^{2m}}{2^{2m+n} m!(n+m)!}, \quad n = 1, 2, 3, \ldots, \tag{30}$$

which is an oscillatory, but not periodic, function.
Question: What should $\sqrt{\lambda}$ be to satisfy $J_n(\sqrt{\lambda} R) = 0$?

Bessel's Equation - Part II

We must now apply the boundary condition. That is, we must require,

$$J_n(\sqrt{\lambda}R) = 0, \tag{31}$$

and find $\sqrt{\lambda}$. However, since the Bessel function is not periodic we have no clear answer to this. It is known that there are infinitely many roots to this special function and finding them requires working with its infinite series definition. Let's just say that we have a set of constants α_{nm} such that,

$$J_n(\alpha_{nm}) = 0, \tag{32}$$

which gives $\sqrt{\lambda}_{nm} = \alpha_{nm}/R$.

General Solution : Fourier-Bessel Series

We know G, Q, W as parametrized by n and m and thus have enough to write down the general solution to this PDE as,

$$u(r,\theta,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[A_{mn}\cos(c\sqrt{\lambda_{mn}}t) + B_{mn}\sin(c\sqrt{\lambda_{mn}}t)\right]\times$$

$$\times J_n(\sqrt{\lambda_{mn}}r)\cos(n\theta)+$$

$$+\left[A^*_{mn}\cos(c\sqrt{\lambda_{mn}}t) + B^*_{mn}\sin(c\sqrt{\lambda_{mn}}t)\right]\times$$

$$\times J_n(\sqrt{\lambda_{mn}}r)\sin(n\theta)$$

Key Point: The time oscillations are the same as the wave equation in (x, y) however now the spatial shape is being controlled by Bessel functions. This can be used to imply that the fundamental shapes allowed on a circular geometry are somehow different than a rectangular geometry.

Fourier-Bessel Series

To finish the problem we must find the coefficients. To do this, we recall the known orthogonality relation,

$$\langle J_n(rk_{n,m}), J_n(rk_{n,i})\rangle = \int_0^R rJ_n(rk_{n,m})J_n(rk_{n,i})dr \tag{33}$$

$$= \frac{\delta_{mi}}{2}\left[RJ_{n+1}(k_{nm}R)\right]^2 . \tag{34}$$

which shows that the coefficients in the Fourier-Bessel series,

$$f(r) = \sum_{m=1}^{\infty} a_m J_n(k_{n,m}r), \tag{35}$$

are given by,

$$a_i = \frac{2}{R^2 J_{n+1}^2(k_{n,m}R)} \int_0^R rJ_n(k_{n,i}r)f(r)dr, \quad i = 1, 2, 3, \ldots . \tag{36}$$

Initial Conditions and Coefficients - Part I

The first initial condition gives,

$$u(r,\theta,0) = f(r,\theta) \tag{37}$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} J_n(\sqrt{\lambda_{mn}}r)\cos(n\theta) + A^*_{mn} J_n(\sqrt{\lambda_{mn}}r)\sin(n\theta) \tag{38}$$

$$= \sum_{n=1}^{\infty} B_n(r)\cos(n\theta) + B^*_n(r)\sin(n\theta), \tag{39}$$

where

$$K_n(r) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(r,\theta)\cos(n\theta)d\theta, \tag{40}$$

$$K^*_n(r) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(r,\theta)\sin(n\theta)d\theta. \tag{41}$$

Initial Conditions and Coefficients - Part II

However, we also have,

$$K_n(r) = \sum_{m=1}^{\infty} A_{mn} J_n(\sqrt{\lambda_{mn}} r), \tag{42}$$

$$K_n^*(r) = \sum_{m=1}^{\infty} A_{mn}^* J_n(\sqrt{\lambda_{mn}} r). \tag{43}$$

Using the orthogonality relations we have,

$$A_{mn} = \frac{2}{R^2 J_{n+1}^2(\sqrt{\lambda_{mn}} R)} \int_0^R r J_n(r\sqrt{\lambda_{nm}}) K_n(r) dr, \tag{44}$$

$$A_{mn}^* = \frac{2}{R^2 J_{n+1}^2(\sqrt{\lambda_{mn}} R)} \int_0^R r J_n(r\sqrt{\lambda_{nm}}) K_n^*(r) dr. \tag{45}$$

Initial Conditions and Coefficients - Part III

Lastly, we have,

$$A_{mn} = \frac{2}{\pi R^2 J_{n+1}^2(\sqrt{\lambda_{mn}}R)} \int_{-\pi}^{\pi}\int_0^R rJ_n(r\sqrt{\lambda_{nm}})(r)f(r,\theta)\cos(n\theta)drd\theta,$$

$$A_{mn}^* = \frac{2}{R^2 J_{n+1}^2(\pi\sqrt{\lambda_{mn}}R)} \int_{-\pi}^{\pi}\int_0^R rJ_n(r\sqrt{\lambda_{nm}})f(r,\theta)\sin(n\theta)drd\theta.$$

Similarly, the second initial condition implies,

$$B_{mn} =$$

$$= \frac{2}{c\sqrt{\lambda_{mn}}\pi R^2 J_{n+1}^2(\sqrt{\lambda_{mn}}R)} \int_{-\pi}^{\pi}\int_0^R rJ_n(r\sqrt{\lambda_{nm}})(r)g(r,\theta)\cos(n\theta)drd\theta,$$

$$B_{mn}^* =$$

$$= \frac{2}{c\sqrt{\lambda_{mn}}\pi R^2 J_{n+1}^2(\sqrt{\lambda_{mn}}R)} \int_{-\pi}^{\pi}\int_0^R rJ_n(r\sqrt{\lambda_{nm}})g(r,\theta)\sin(n\theta)drd\theta.$$

Conclusions - Part II

At this point we have the following conclusions:

1. When dealing with a PDE the coordinated system should be chosen according to the boundary geometry.
2. Doing so changes the form that the Laplacian takes.
3. The Laplacian is a manifestly self-adjoint differential operator and leads to Sturm-Liouville problems.
4. SL problems lead to a complete set of orthogonal basis functions.
5. The orthogonal basis functions controls the shape of the unknown function is space-time.
6. In different geometries you can get different shapes and some of these shapes can be inherently more complicated than in other geometries.