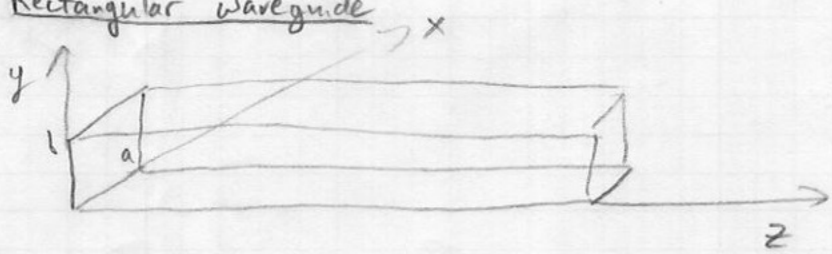
# Lecture 11: Waveguides that can actually exist

So that parallel plate waveguide is good for getting the basics down, but its not the most common object in real life. We're going to look at rectangular + coax waveguides, which <u>are</u> a bit more common in real life.

## Rectangular waveguide



Ok, so we have a really long metallic box, of x-width a and y-width b.

As usual, we'll take E + B to be zero inside the metal.

TE modes: As before, we construct TE solutions by assuming a form that's, well, transverse:

$$\vec{E} = \left[E_x \hat{\imath} + E_y \hat{\jmath}\right] e^{i(kz-\omega t)}$$

Also as usual, we start hitting the trial solution with Maxwell eqns until its legit.

$$\nabla \cdot \vec{E} = 0 \Rightarrow \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y}\right) e^{i(kz-\omega t)} = 0$$

So keep in mind we're constructing a solution, so we can do pretty much whatever helps. Here's a trick.

Define some function $\psi(x,y)$ and build $E_x$ + $E_y$ by taking

$$E_x = -\partial\psi/\partial y \qquad E_y = \partial\psi/\partial x$$

That'll force $\frac{\partial E_x}{\partial x} + \partial E_y/\partial y = 0$

So $$\vec{E} = \left(-\partial\psi/\partial y \, \hat{\imath} + \frac{\partial\psi}{\partial x}\hat{\jmath}\right) e^{i(kz-\omega t)}$$

Next we'll crank this through the wave equation for $\vec{E}$

$$\nabla^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

The book conveniently omits the algebra that follows. Its actually pretty damned long. But eventually you can get to

$$\nabla^2 \psi - k^2 \psi + \frac{\omega^2}{c^2}\psi = 0$$

$$\Rightarrow \quad \nabla^2 \psi = -\gamma^2 \psi \qquad \text{with } \gamma = \sqrt{\frac{\omega^2}{c^2} - k^2}$$, much like before

Now, $\psi$ is a function of $x$ & $y$, so this separates into

$$\psi = f(x)g(y) \qquad \text{with} \quad \frac{1}{f}\frac{d^2 f}{dx^2} + \frac{1}{g}\frac{d^2 g}{dy^2} = -\gamma^2$$

$$\Rightarrow \quad \frac{1}{f} f'' = -\mu^2 \,, \quad \frac{1}{g} g'' = -\nu^2 \qquad \text{with} \quad \mu^2 + \nu^2 = \gamma^2$$

$$\Rightarrow \quad f(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$$

$$g(y) = c_3 \cos(\nu y) + c_4 \sin(\nu y)$$

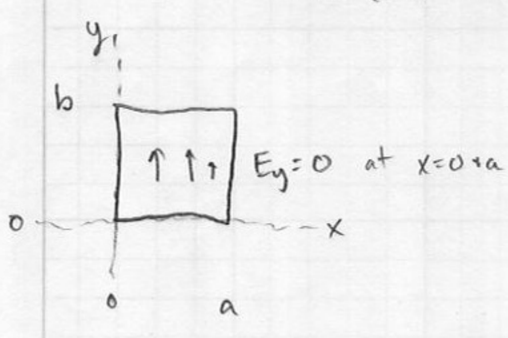
Remember: the product of these only gives you $\psi$, not $E$ or $B$.

Let's apply some BCs. $E_{\parallel}$ is always continuous, so

$E_y = 0$ at $x=0$ and $x=a$. And $E_y = \frac{\partial \psi}{\partial x} = f'g$

So if $f'g = 0$ for any $y$, (when $x=0$ or $a$) we need $f'(0)=0$ and $f'(a)=0$

$$f' = -\mu c_1 \sin(\mu x) + \mu c_2 \cos(\mu x)$$




To be 0 at $x=0$ we need $c_2 = 0$, so

$$f'(a) = -\mu c_1 \sin(\mu a)$$

For that to be 0, we need

$$\boxed{\mu = \frac{m\pi}{a}}$$

The same kind of reasoning with $E_x$ will yield $\boxed{\nu = \frac{n\pi}{b}}$ and $c_4 = 0$

So $\gamma = \sqrt{\frac{\omega^2}{c^2} - k^2} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$

And $\psi = \psi_0 \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)$

$$\Rightarrow \vec{E}_{TE_{m,n}} = \psi_0 \left[ \frac{n\pi}{b} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \hat{\imath} - \frac{m\pi}{a} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \hat{\jmath} \right] e^{i(kz - \omega t)}$$

This we can recognize as basically a plane wave in z on top of standing waves in x & y, with the mode m,n tied to how many wavelengths fit in the box. Note (0,0) can't exist.

We can construct $\vec{B}$ now via $\nabla \times \vec{E} = -\partial \vec{B}/\partial t$. The steps are boring, but there's an interesting bit of physics in the answer, so I'll write it down:

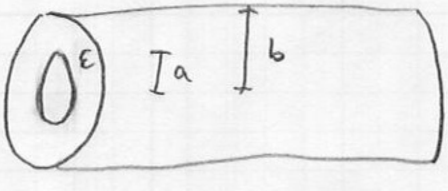
$$\vec{B}_{TE_{m,n}} = \frac{k\psi_0}{\omega} \left[ \frac{m\pi}{a} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \hat{\imath} + \frac{n\pi}{b} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \hat{\jmath} + \frac{i\gamma^2}{k} \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \hat{k} \right] e^{i(kz - \omega t)}$$

$B_z$ is the only part of either B or E that depends on $\lambda$ (via $1/k$). So as $\lambda$ gets small, so does $B_z$. As $\lambda$ gets small, the wave approaches a wave that is transverse in both E + B, which doesn't normally exist in this kind of waveguide.

Interesting question: As $\lambda$ gets really small, does our solution approach something that could exist in free space? Should it?

(discussion?)

## A new coordinate system: The coax cable




Imagine taking a parallel-plate waveguide and rolling it up. It'll give you a coax cable with free space from $a < r < b$, or if you prefer, some dielectric ε.

I put things in terms of rolling up a parallel plate waveguide because it motivates trying to construct a solution from the parallel plate solutions. In particular, we're going to try to contruct a TEM solution that fits this geometry.


Let's try $\quad \vec{E} = E(r)\, e^{i(kz-\omega t)}\, \hat{r}$

$$\vec{B} = B(r)\, e^{i(kz-\omega t)}\, \hat{\phi}$$

As is typical, we take our fundamental solutions & start banging on them with Maxwell's equations, but this time with operators expressed in cylindrical.

$\nabla \cdot \vec{B}$ is definitely zero, since $B$ is a function of $r$ but is in the $\phi$ direction

Let's try $\nabla \cdot \vec{E} = 0$:

$$\nabla \cdot \vec{E} = \frac{1}{r}\frac{\partial}{\partial r}\left( r \cdot E(r)\, e^{i(kz-\omega t)} \right) = 0$$

For this to be true, we must have $rE$ = constant, so

$$E(r) = C/r$$

Two down, two to go. Let's do Faraday's law:

$$\nabla \times \vec{E} = ik E(r)\, e^{i(kz-\omega t)}\, \hat{\phi} = -\partial \vec{B}/\partial t = i\omega B(r)\, e^{i(kz-\omega t)}\, \hat{\phi}$$

$$\Rightarrow \quad kE(r) = \omega B(r) \quad \Rightarrow \quad B(r) = \frac{kC}{\omega r} \qquad (1)$$

Next, Ampere's law

$$\nabla \times \vec{B} = ik B(r)\, e^{i(kz-\omega t)}\, \hat{r} = \mu_0 \varepsilon\, \partial \vec{E}/\partial t = i\mu_0 \varepsilon \omega\, E(r)\, e^{i(kz-\omega t)}\, \hat{r}$$

$$\Rightarrow \quad kB = \mu_0 \varepsilon \omega E \quad \Rightarrow \quad B(r) = \frac{\mu_0 \varepsilon \omega C}{kr} \qquad (2)$$

Comparing 1 + 2, we see $\frac{K}{\omega} = \frac{\mu_0 \varepsilon \omega}{K} \Rightarrow K^2 = \frac{\omega^2}{v^2}$

Giving us the dispersion relation $\boxed{K = \omega / v}$

Okay, so let's try to fix C. In practice coax cables have some voltage applied to them (or rather, a voltage difference applied between the inner & outer bits). In general,

$$V = \int \vec{E} \cdot d\vec{l}, \quad V = \int_a^b E\,dr = \int_a^b \frac{C}{r}\,dr \Rightarrow V = C \ln(b/a)$$

$$\Rightarrow C = V / \ln(b/a)$$

$$\Rightarrow \quad \vec{E} = \frac{V_0}{r \ln(b/a)} e^{i(kz - \omega t)} \hat{r}$$

$$\vec{B} = \frac{V_0}{v r \ln(b/a)} e^{i(kz - \omega t)} \hat{\phi}$$

What about the BCs? $B, E = 0$ at $r = a, b$

And $E_{\parallel}$, $B_{\perp}$ are continuous, but $B_{\perp}$ + $E_{\parallel}$ are zero everywhere so that's kind of free

Discontinuities in $E_{\perp}$, $B_{\parallel}$ yield surface charges & currents.

(picture)