10-26-07

Note Title 10/25/2007

Fourièr Transforms

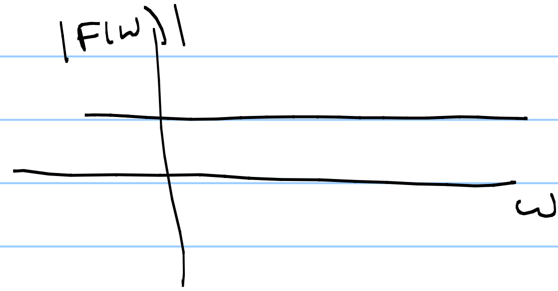
$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} \, d\omega$$

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt$$

ex. $f(t) = \delta(t)$

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} \delta(t) e^{-i\omega t} \, dt}$$

$$= e^{-i\omega t} \Big|_{t=0} = 1$$



$f(t) = \delta(t - t')$

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} \delta(t-t') e^{-i\omega t} \, dt}$$

$$e^{-i\omega t} \Big|_{t=t'}$$

$$F(\omega) = \frac{1}{\sqrt{2\pi}} e^{-i\omega t'}$$

$$Re[F(\omega)] = \frac{1}{\sqrt{\ }} \cos\omega t'$$

$$Im[F(\omega)] = \frac{1}{\sqrt{\ }} \sin \omega t$$

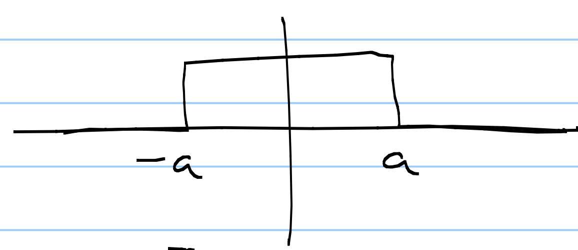
$$|F(\omega)|^2 = \frac{1}{\ }$$

<u>important point</u>

Fourier Transform of $\delta(t-t')$
$= 1$ if and only if $t'=0$

<u>But</u> power spectrum of $\delta(t-t')$
is alway $\frac{1}{\ }$

Ex. $f(x) =$



$$\begin{cases} 1 & \text{if } -a \le x \le a \\ 0 & \text{otherwise} \end{cases}$$

$x \Leftrightarrow k \quad t \Leftrightarrow \omega$

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-ikx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{-ik} e^{-ikx} \Big|_{-a}^{a} \right]$$

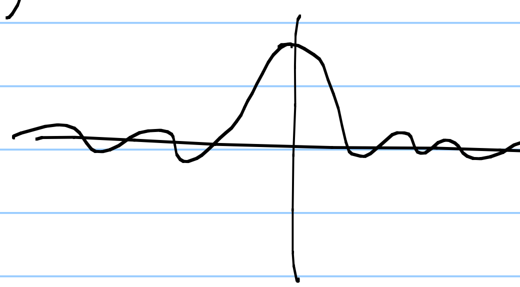
$\cos(-x) = \cos x$

$\sin(-x) = -\sin(x)$

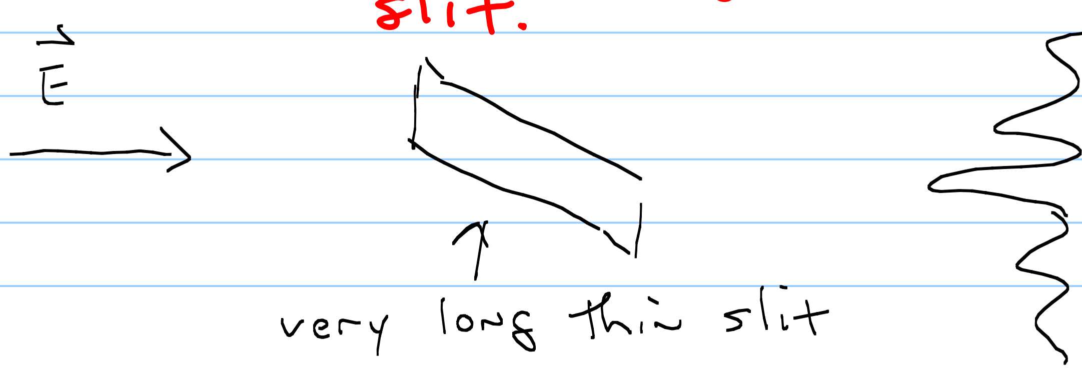
$$= \frac{i}{\sqrt{2\pi}\, k} \left[(\cos ka - i \sin ka) - (\cos ka + i \sin ka) \right]$$

$$= \frac{i}{\sqrt{2\pi}\, k} \left(-2i \sin ka \right)$$

$$\frac{2}{\sqrt{2\pi}} \; \frac{\sin ka}{k}$$

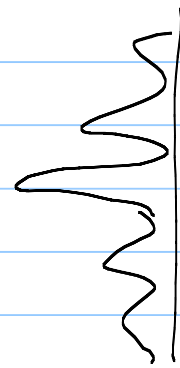


Claim: $|F(k)|^2$ is the diffraction pattern of a rectangular slit.



very long thin slit

plot $\quad \dfrac{\sin^2 ka}{k^2}$

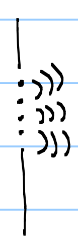


becomes k axis in the experiment

physics $\quad$ screen obstructs 100% of the light except for the slit. Imagine each point in the slit is a source of secondary wavelets

— Huyghen's principle

mathematics

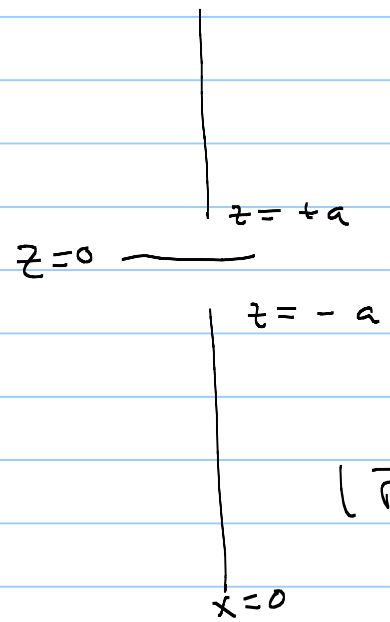


each little point source is a spherical wave

$$\frac{e^{ik(r-r')}}{|r-r'|}$$

So field on the screen is the superposition of all of those point sources

$\vec{E}$ field at a point r' (on screen) is



$$\int_{-a}^{a} \frac{e^{ik(r-r')}}{|r-r'|}\, dz$$

$$|\vec{r}-\vec{r}'| = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$$

if the slit is small and the distance to the screen is large, then the integral can be approximated. [see references on the wiki] Far field approx is called **Fraunhoffer Diffraction.**

Fourier Transform of a Gaussian

Lemma $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$

Proof Let $\displaystyle I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$

$$I^2 = \left[\int_{-\infty}^{\infty} e^{-x^2}\,dx\right]\underbrace{\left[\int_{-\infty}^{\infty} e^{-y^2}\,dy\right]}_{\text{dummy var. of integ.}}$$

recast as double integral

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy$$

$$= \int_0^{\infty}\int_0^{2\pi} e^{-r^2}\,r\,dr\,d\theta$$

$$= 2\pi \int_0^{\infty} e^{-r^2}\,r\,dr$$

Let $\rho = r^2$ $\quad d\rho = 2r\,dr$

$$I^2 = 2\pi \int_0^{\infty} e^{-\rho}\,\frac{d\rho}{2}$$

$$= \pi\left[-e^{-\rho}\Big|_0^{\infty}\right] = \pi$$

$$\Rightarrow I = \sqrt{\pi}$$