Some simple examples from plasma physics

model plasma - free electron gas

electrons free to move, no collisions, no resistance

neutralizing background of ions

charge density $\rho_e = -e n_e$ $\quad n_e$ = number density cm^{-3} in lab units

Debye shielding

cold plasma; like metals, prevents the penetration of E field

- charges arrange to cancel the field

$\oplus$ positive test charge accumulates -'ve charge

"warm" plasma: balance mean thermal energy w/ el. pot'l

in equilibrium, use Boltzmann distribution

reminder:

$$f(v) = A e^{-(\frac{1}{2}mv^2)/kT}$$

(one dimension) $\quad$ no potential, constant density

number density:

$$n = \int_{-\infty}^{\infty} f(v)\, dv \rightarrow A = n \left(\frac{m}{2\pi kT}\right)^{1/2}$$

Add a potential $U(x)$

$$f(v) = A \exp\left[-\left(\tfrac{1}{2}mv^2 + U(x)\right)/kT\right]$$

$$\text{now } n(x) = A \int e^{-U(x)/kT} e^{-\frac{1}{2}mv^2/kT} dv$$

pick a reference pt at $x_0 \rightarrow n_{ref}$

$$n(x) = n_{ref}\, e^{-U/kT}$$

example - for gravity $U = mgx$

→ exponential fall off in density w/ altitude.

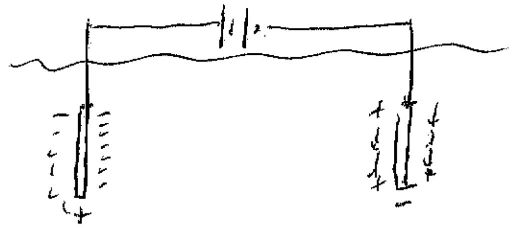
electric pot'l $U = q\phi = -e\phi$

pick ref at $x = \infty$ (far from test charge)

$$n(x=\infty) = n_\infty$$

$$\rightarrow \quad n_e(x) = n_\infty e^{e\phi/kT_e}$$

consider a grid at a fixed pot'l (one-dim)



solve for $\phi(x) \rightarrow n_e(x)$

$$\nabla \cdot \vec{E} = 4\pi\rho = -4\pi e(n_e - n_i)$$

$$\vec{E} = -\nabla\phi$$

→ poisson eqn $\nabla^2\phi = 4\pi e(n_e - n_i)$

assume ions are fixed:

1-D $$\frac{d^2\phi}{dx^2} = 4\pi e n_\infty \left(e^{e\phi/kT_e} - 1\right)$$

where $e\phi/kT_e \ll 1$, expand expon.

$$e^x \approx 1 + x + \frac{1}{2}x^2 \dots$$

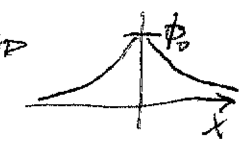
keep lowest order term

$$\frac{d^2\phi}{dx^2} = \underbrace{\frac{4\pi n_\infty e^2}{kT_e}}_{\text{dimensions } m^{-2}} \phi$$

define $\lambda_D^{-1} = \left(\frac{4\pi n_e e^2}{kT_e}\right)^{1/2}$ $\quad n_e$ = background density

$$\rightarrow \quad \frac{d^2\phi}{dx^2} = \lambda_D^{-2}\,\phi \qquad + \text{ sign} \rightarrow \text{exponential solutions}$$

$$\phi = \phi_0\, e^{\pm x/\lambda_D} = \phi_0\, e^{-|x|/\lambda_D}$$



calculate λ_D = Debye length

gaussian $\quad 4\pi e^2 \rightarrow$ SI $\quad e^2/\varepsilon_0$

or $\quad e^2 \rightarrow \frac{1}{4\pi\varepsilon_0} e^2$

$$\lambda_D = \left(\frac{\varepsilon_0 k_B T_e}{n_e e^2}\right)^{1/2}$$

dimension check: $\quad k_B T_e \sim$ energy

e^2/r = energy $\rightarrow e^2 \sim$ energy·len

$$\lambda_D \sim \left(\frac{\text{energy}}{(\text{len})^{-3}\,\text{energy}\cdot\text{len}}\right)^{1/2} \sim \text{len.}$$

experimental units:

$k_B T_e \sim eV$

$n_e \sim cm^{-3}$

$\rightarrow \lambda_D = 743 \left(\frac{k_B T_e}{n_e}\right)^{1/2} cm \qquad k_B T_e$ in eV, $\quad n_e$ in cm^{-3}

at 1 Torr (760 Torr at STP), room temp

ideal gas $\quad PV = N k_B T \rightarrow n = P/k_B T$

$k_B T \sim 1/40$ eV

$n \sim 3 \times 10^{16}\ cm^{-3}$

full ionization, $\quad k_B T_e = 1$ eV

$\rightarrow \lambda_D \sim 40$ nm

calc. # particles in sphere of radius λ_D:

$N_D = n_e \cdot \frac{4}{3}\pi\lambda_D^3 \sim 10$

if $N_D \gg 1 \rightarrow$ <u>collective</u> behavior

"Cyclotron" frequency:

moving electron with $\vec{V}_e \perp \vec{B}$ has circular path

$$\vec{F} = q\vec{E} + q\frac{\vec{V}}{c} \times \vec{B} \qquad \text{gaussian}$$

HW: solve for eqn.

scaling $\quad F = m\frac{dv}{dt} \sim e\frac{v}{c}B_o$

$$\rightarrow \text{frequency} \sim \frac{eB_o}{m_e c} \equiv \omega_c$$

go to eqn to figure units:

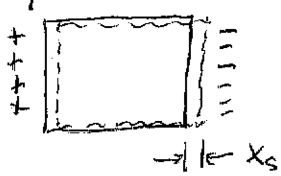
SI $\quad F \sim -e\vec{v} \times \vec{B}$

$\therefore$ take B_o/c (gaussian) $\rightarrow$ B_o (SI)

then use B_o in tesla, e in coulombs.

Plasma frequency

consider slab of plasma, with all electrons displaced by X_s



net $\vec{E} = 0$ outside, $E_{inside} \propto X_s$ (use Gauss' law)

$\rightarrow$ spring like restoring force

electrons collectively oscillate at

$$\omega_p = \left(\frac{4\pi n_e e^2}{m_e}\right)^{1/2} \equiv \text{plasma frequency}$$

$$\text{SI} \quad \omega_p = \left(\frac{n_e e^2}{\varepsilon_o m_e}\right)^{1/2} = 5.7\times10^4\, n_e^{1/2} \text{ rad/s} \quad n_e \text{ in cm}$$

at 1 Torr $\quad \omega_p \sim 10^{13}$ rad/s, $\nu = \frac{\omega_p}{2\pi} \sim 1$ THz.

Energy density in fields

mechanics: $\uparrow h$ $PE = mgh$.

EM: $+ -$ $\rightarrow$ $+ \overset{\longleftarrow l \longrightarrow}{} -$ create E field

Field theory: fields are real in the sense that they have energy, momentum (linear and angular)

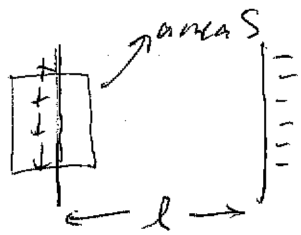
fields are distributed, so work with <u>densities</u>

e.g. u_E = energy density in E field.

review:

charge a capacitor $\quad W = \frac{1}{2} C V^2$

Gauss' law $\rightarrow$ field



$$\int \vec{E} \cdot d\vec{a} = \int 4\pi \rho \, d^3x$$

$$\Rightarrow E_L \cdot 2S = 4\pi q$$

$$E_L = \frac{2\pi q}{S}, \quad E_R = \frac{2\pi q}{S}$$

add both $\quad q = \frac{ES}{4\pi} \quad$ field is constant inside

$$\vec{E} = -\nabla \phi, \text{ so } \quad E = \Delta\phi / l \quad (= V/l)$$

$$\therefore q = \Delta \phi \frac{S}{4\pi l} \quad \text{and} \quad C \equiv q/\Delta\phi = \frac{S}{4\pi l}$$

with dielectric, $\nabla \cdot \vec{D} = \nabla \cdot \epsilon \vec{E} = 4\pi\rho$

$$\rightarrow C = \frac{\epsilon}{4\pi} \frac{S}{l}$$

Work done to charge capacitor

$$F = qE \qquad \Delta W = \Delta q\, E\, l$$

$$W = l\int E(q')\,dq' = l\cdot\int_0^q \frac{4\pi q'}{S}\,dq' = \frac{1}{C}\frac{1}{2}q^2 = \frac{1}{2}CV^2$$

in terms of field: $\qquad q = CV$

$$C = \frac{\varepsilon}{4\pi}\frac{S}{l} \qquad V = \Delta\phi = El$$

$$W = \frac{1}{2}\frac{\varepsilon}{4\pi}\frac{S}{l}E^2 l^2 = \frac{1}{8\pi}\varepsilon E^2\cdot Sl \rightarrow \int \frac{\varepsilon}{8\pi}E^2\,d^3x$$

<u>energy density</u> $U_E \equiv \dfrac{\vec{D}\cdot\vec{E}}{8\pi} \left(= \dfrac{|E|^2}{8\pi} \text{ in vacuum}\right)$

For magnetic field, consider an inductor: (see Pollack 10.5)

back EMF while current is changing

$$\mathcal{E} = -\frac{1}{c}L\, dI/dt$$

power to push current I against $\mathcal{E}$ $(P = IV)$

$$P = -I\mathcal{E} = +IL\, dI/dt$$

$$\rightarrow W = \frac{1}{2}LI^2 \text{ energy stored.}$$

relate to fields

$$\mathcal{E} \equiv -\frac{1}{c}L\frac{dI}{dt} \equiv -\frac{d\Phi_m}{dt}, \qquad \Phi_m \overset{c\cdot}{=} LI = \int_{A \text{ of loop}} \vec{B}\cdot d\vec{a}$$

$$\vec{B} = \nabla\times\vec{A} \rightarrow \qquad = \oint_L \vec{A}\cdot d\vec{l}$$

$$\therefore W = \frac{1}{2}LI^2 = \frac{1}{2}\frac{1}{c}\oint I\vec{A}\cdot dl = \frac{1}{2c}\int \vec{J}\cdot\vec{A}\,d^3x$$

$$\text{now } \nabla\times\vec{H} = \cancel{\frac{1}{c}\frac{\partial\vec{D}}{\partial t}} + \frac{4\pi}{c}\vec{J}$$

0 ~~if~~ no $\vec{D}$

So $\frac{c}{4\pi}(\nabla \times \vec{H}) \cdot \vec{A} = \frac{c}{4\pi}\left[\nabla \cdot (\vec{H} \times \vec{A}) + \vec{H} \cdot (\nabla \times \vec{A})\right]$

now $\int \nabla \cdot (\vec{H} \times \vec{A})\, d^3x = \int_{\partial S} (\vec{H} \times \vec{A}) \cdot d\vec{S}$

$\to 0$ for a localized field.

So $W = \frac{1}{2c} \cdot \frac{c}{4\pi} \int \vec{H} \cdot \vec{B}\, d^3x = \int \underbrace{\frac{H \cdot B}{8\pi}}_{u_B}\, d^3x$