MATH348-Advanced Engineering Mathematics		Ad Hoc Homework One
	Differential Equations	
Text: 12.3, 12.5	Lecture Notes: 13-15	Lecture Slides: 6

Quote of Ad Hoc Homework One	
Don't believe the florist when he tells you the roses are free.	
	Ween : Roses are Free (1994)

1. ODE Review

When solving the linear wave equation, heat equation and Poisson's partial differential equation (PDE), on compact domains of $\mathbb{R}^n$ separation of variables is the typical method. When using separation of variables one trades a PDE for a class of ordinary differential equations (ODE) that manifest a set of orthogonal functions that can be used to represent the solution to the original PDE. For most of our work we will concentrate on,

$$y'' + \lambda y = 0,\ \lambda \in \mathbb{R},\ x \in (0, L). \tag{1}$$

1.1. **General Solution to the ODE.** Justify that the following functions solve the ODE for particular values of λ and arbitrary constants $c_i \in \mathbb{R}$ for $i = 1, 2, 3, 4, 5, 6$.[1]

Case	Function 1	Function 2
$\lambda > 0$	$y(x) = c_1 \cos(\sqrt{\lambda}x)$	$y(x) = c_2 \sin(\sqrt{\lambda}x)$
$\lambda < 0$	$y(x) = c_3 \cosh(\sqrt{\lvert\lambda\rvert}x)$	$y(x) = c_4 \sinh(\sqrt{\lvert\lambda\rvert}x)$
$\lambda = 0$	$y(x) = c_5$	$y(x) = c_6 x$

2. BVP Overview

Boundary value problems (BVP) typically arise within the context of PDE, which are equations modelling the evolution of a quantity in both space and time. There are important general results for BVP, which are set within the context of Sturm-Liouville problems.[2] What can be efficiently done by hand tends to be limited. The problem, in Cartesian coordinates, is to find all solutions to,

$$y'' + \lambda y = 0,\ \lambda \in \mathbb{R},\ x \in (0, L), \tag{2}$$

which also satisfy,

$$l_1 y(0) + l_2 y'(0) = 0, \tag{3}$$

$$r_1 y(L) + r_2 y'(L) = 0. \tag{4}$$

This problem is intractable, by hand, for general values of l_1, l_2, r_1, r_2. However, the following set of values,

	l_1	l_2	r_1	r_2
Case I	1	0	1	0
Case II	0	1	0	1
Case III	1	0	0	1
Case IV	0	1	1	0

lead to BVP that can be solved by hand.

2.1. **Application of Boundary Conditions.** We have seen Case I in long homework 2 and Case II in long homework 3. Now we concentrate on Case III and Case IV. Before you begin you may want to collect the previous results so that you have them all on one page.

2.1.1. *Case III.* From the previous table of functions, first show that $y(0) = 0$ implies that $c_1 = c_3 = c_5 = 0$. Next show that $y'(L) = 0$ implies that $c_4 = c_6 = 0$. This leaves just the sine function to deal with. Lastly, show that $y(x) = c_2 \sin(\sqrt{\lambda}x)$ satisfies the condition $y'(L) = 0$ for the specific values $\sqrt{\lambda} = (2n+1)\frac{\pi}{2L}$ where $n = 1, 2, 3, \ldots$.

[1] There are, of course, many ways to do this. You could re-derive the given information, quote a previous homework or substitute the solutions into the ODE. You choice.

[2] Some of these results can be found in long homework number 7

2.1.2. *Case IV.* From the previous table of functions, first show that $y'(0) = 0$ implies that $c_2 = c_4 = c_6 = 0$. Next show that $y(L) = 0$ implies that $c_3 = c_5 = 0$. This leaves just the cosine function to deal with. Lastly, show that $y(x) = c_1 \cos(\sqrt{\lambda} x)$ satisfies the condition $y(L) = 0$ for the specific values $\sqrt{\lambda} = (2n+1)\frac{\pi}{2L}$ where $n = 1, 2, 3, \ldots$.

3. Power-Series Solutions to ODE's and Hyperbolic Trigonometric Functions

Consider the ordinary differential equation:

$$y'' - y = 0 \tag{5}$$

3.1. **General Solution - Standard Form.** Show that the solution to (5) is given by $y(x) = c_1 e^x + c_2 e^{-x}$.

3.2. **General Solution - Nonstandard Form.** Show that $y(x) = b_1 \sinh(x) + b_2 \cosh(x)$ is a solution to (5) where $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$ and $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$.

3.3. **Conversion from Standard to Nonstandard Form.** Show that if $c_1 = \dfrac{b_1 + b_2}{2}$ and $c_2 = \dfrac{b_1 - b_2}{2}$ then $y(x) = c_1 e^x + c_2 e^{-x} = b_1 \cosh(x) + b_2 \sinh(x)$.

3.4. **Relation to Power-Series.** Assume that $y(x) = \displaystyle\sum_{n=0}^{\infty} a_n x^n$ to find the general solution of (5) in terms of the hyperbolic sine and cosine functions. [3]

4. Heat Equation on a closed and bounded spatial domain of $\mathbb{R}^{1+1}$

Consider the one-dimensional heat equation,

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad , \tag{7}$$

$$x \in (0, L)\,, \qquad t \in (0, \infty)\,, \qquad c^2 = \frac{K}{\sigma \rho}. \tag{8}$$

Equations (12)-(13) model the time-evolution of the temperature, $u = u(x,t)$, of a heat conducting medium in one-dimension. The object, of length L, is assumed to have a homogenous thermal conductivity K, specific heat σ, and linear density ρ. That is, $K, \sigma, \rho \in \mathbb{R}^+$. If we consider an object of finite-length, positioned on say $(0, L)$, then we must also specify the boundary conditions[4],

$$u_x(0,t) = 0, u_x(L,t) = 0, . \tag{9}$$

Lastly, for the problem to admit a unique solution we must know the initial configuration of the temperature,

$$u(x, 0) = f(x). \tag{10}$$

4.1. **Separation of Variables : General Solution.** Assume that the solution to (12)-(13) is such that $u(x,t) = F(x)G(t)$ and use separation of variables to find the general solution to (12)-(13), which satisfies (14)-(15). [5]

4.2. **Qualitative Dynamics.** Describe how the long term behavior of the general solution to (12)-(15) changes as the thermal conductivity, K, is increased while all other parameters are held constant. Also, describe how the solution changes when the linear density, ρ, is increased while all other parameters are held constant.

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[3] The hyperbolic sine and cosine have the following Taylor's series representations centred about $x = 0$,

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \qquad \sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}. \tag{6}$$

It is worth noting that these are basically the same Taylor series as cosine/sine with the exception that the signs of the terms do not alternate. From this we can gather a final connection for wrapping all of these functions together. If you have the Taylor series for the exponential function and extract the even terms from it then you have the hyperbolic cosine function. Taking the hyperbolic cosine function and alternating the sign of its terms gives you the cosine function. Extracting the odd terms from the exponential function gives the same statements for the hyperbolic sine and sine functions. The reason these functions are connected via the imaginary number system is because when i is raised to integer powers it will produce these exact sign alternations. So, if you remember $e^x = \sum_{n=0}^{\infty} x^n/n!$ and $i = \sqrt{-1}$ then the rest (hyperbolic and non-hyperbolic trigonometric functions) follows!

[4] Here the boundary conditions correspond to perfect insulation of both endpoints

[5] An insulated bar is discussed in examples 4 and 5 on page 557.

4.3. **Fourier Series : Solution to the IVP.** Define,

$$f(x) = \begin{cases} \dfrac{2k}{L}x, & 0 < x \leq \frac{L}{2}, \\ \\ \dfrac{2k}{L}(L-x), & \frac{L}{2} < x < L \end{cases} \tag{11}$$

and for the following questions we consider the solution, u, to the heat equation given by, (12)-(13), which satisfies the initial condition given by (16). [6] For $L = 1$ and $k = 1$, find the particular solution to (12)-(13) with boundary conditions (14)-(15) for when the initial temperature profile of the medium is given by (16). Show that $\lim_{t\to\infty} u(x,t) = f_{avg} = 0.5$.[7]

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[6]When solving the following problems it would be a good idea to go back through your notes and the homework looking for similar calculations.

[7]It is interesting here to note that though the initial condition f doesn't appear to satisfy the boundary conditions its periodic Fourier extension does. That is, if you draw the even periodic extension of the initial condition then you would see that the slope is not well defined at the end points. Remembering that the Fourier series averages the right and left hand limits of the periodic extension of the function f at the endpoints shows that the boundary conditions are, in fact, satisfied, since the derivative of an average is the average of derivatives.

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