Assume $V(s,\varphi) = S(s)\Phi(\varphi)$

2. Derive a general expression for the solution to Laplace's equation in cylindrical coordinates (s and ϕ are the radial, angular coordinates) using separation of variables. Note $\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}(s\frac{\partial V}{\partial s}) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} = 0$.

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial (S\Phi)}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 S\Phi}{\partial \varphi^2} = \frac{1}{s}\Phi\frac{d}{ds}\left(s\frac{dS}{ds}\right) + \frac{S}{s^2}\frac{d^2\Phi}{d\varphi^2} = 0$$

multiply by s & divide by $V = S\Phi$

$$\underbrace{\frac{s}{S}\frac{d}{ds}\left(s\frac{dS}{ds}\right)}_{C_1} + \underbrace{\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}}_{C_2} = 0$$

$C_1 + C_2 = 0$

each term is independent of the other so they must each equal a constant if they are to add to zero

C_2 must be negative or else we get exponential or linear functions which do not come back to the same value if φ changes by 2π. Set $C_2 = -k^2$ then $\frac{d^2\Phi}{d\varphi^2} = -k^2\Phi$ $\boxed{\Phi = A\cos k\varphi + B\sin k\varphi}$

also since $\Phi(\varphi + 2\pi) = \Phi(\varphi)$ k must be an integer $k = 0, 1, 2, \cdots$

The radial eqn becomes $s\frac{d}{ds}\left(s\frac{dS}{ds}\right) = k^2 S$

Assume $S = s^n$

$s\frac{d}{ds}(sns^{n-1}) = ns\frac{d}{ds}(s^n) = n^2 s s^{n-1} = n^2 s^n = k^2 S = k^2 s^n$

So $k = \pm n$ and $\boxed{S = Cs^k + Ds^{-k}}$

We must treat $k=0$ differently. Then $s\frac{d}{ds}\left(s\frac{dS}{ds}\right) = 0$

$\Rightarrow s\frac{dS}{ds} = \text{constant} \Rightarrow \frac{dS}{ds} = \frac{\text{constant}}{s} \Rightarrow dS = \frac{\text{constant}}{s}ds$

Integrate $\Rightarrow$ $\boxed{S = \text{constant}\ln(s) + D}$ another constant

General soln is a superposition of these for all k.