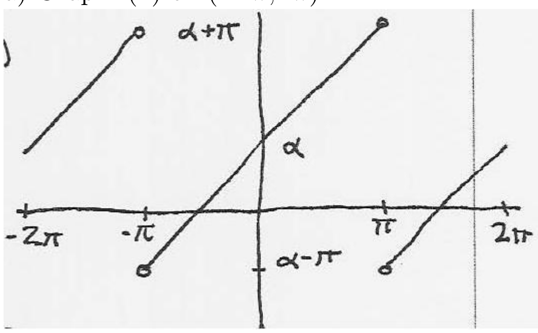
Homework #6 Solutions

1. Consider the function

$$f(x) = x + \alpha \quad \alpha \in R$$

which is 2π periodic on the interval $-\pi \leq x \geq \pi$

a) Graph f(x) on $(-2\pi, 2\pi)$



b) Is the function even, odd, or neither?

neither.

c) Determine the Fourier Coefficients a_0, a_n, b_n

We have that the Fourier Series of f(x) should be the addition of the Fourier series for $f_1(x) = x$ and the Fourier Series for $f_2(x) = \alpha$.

We have from class that

$$f_1(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} sin(nx)$$

Formulas from Kreyscig p. 480.

For $f_2(x)$

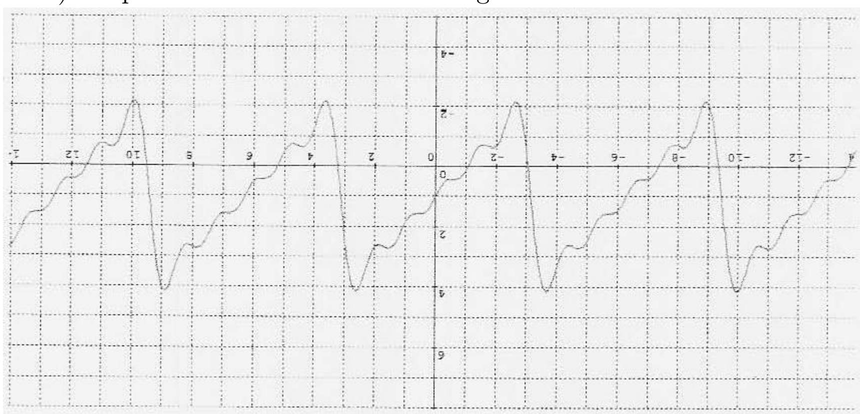
$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \alpha dx = \frac{\alpha x}{2\pi}\Big|_{-\pi}^{\pi} = \alpha \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \alpha cos(nx) dx = \frac{\alpha}{\pi} sin(nx)\Big|_{-\pi}^{\pi} = 0 \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \alpha sin(nx) dx = \frac{\alpha}{\pi} cos(nx)\Big|_{-\pi}^{\pi} = 0
\end{aligned}$$

Thus, for $f(x) = x + \alpha$

$$f(x) = \alpha + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} sin(nx)$$

$$a_0 = \alpha, \quad a_n = 0, \quad b_n = \frac{2(-1)^{n+1}}{n}$$

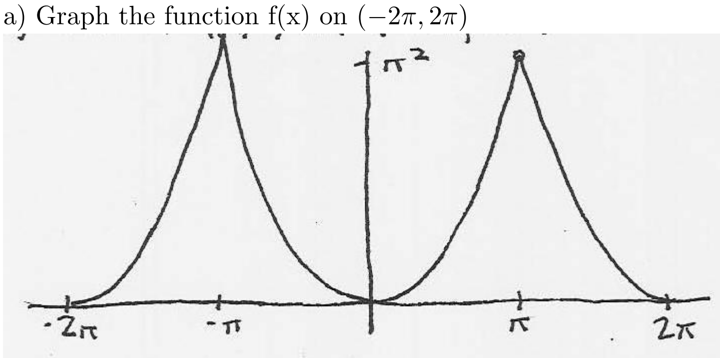
d) Graph the first five terms assuming $\alpha = 1$.



2. Consider the Function

$$f(x) = x^2 \quad x \in [-\pi, \pi]$$

which is 2π periodic
a) Graph the function f(x) on $(-2\pi, 2\pi)$



b) Is the functino even, odd or neither?
even.
c) Determine the Fourier coeffiecients a_0, a_n, b_n.

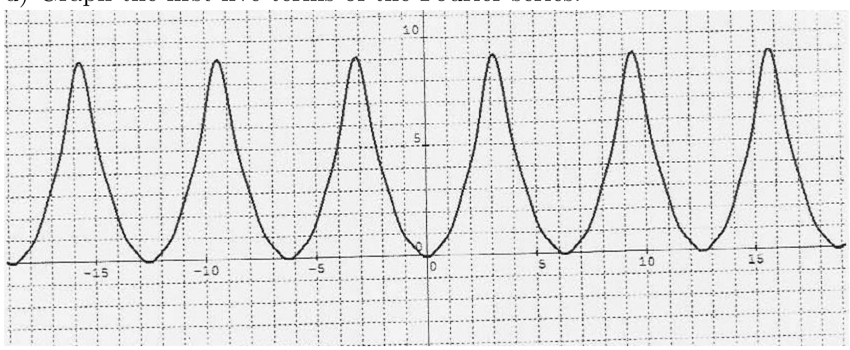
$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 dx \Rightarrow \text{Because it's an even fnx} \Rightarrow \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx \\
&= \left.\frac{x^3}{3\pi}\right|_0^{\pi} = \frac{\pi^2}{3} \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 cos(nx) dx \Rightarrow \text{Because it's an even fxn} \Rightarrow = \frac{2}{\pi}\int_0^{\pi} x^2 cos(nx) dx \\
&= \frac{2}{\pi}\left[\frac{x^2}{n} sin(nx) + \frac{2x}{n^2} cos(nx) - \frac{2}{n^3} sin(nx)\right]_0^{\pi} \\
&= \frac{2}{\pi}\left[\frac{2\pi}{n^2} cos(n\pi)\right] = \frac{4(-1)^n}{n^2}
\end{aligned}$$

$$b_n = 0 \quad \text{Because } f(x) \text{ is even.}$$

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} cos(nx)$$

$$a_0 = \frac{\pi^2}{3} \quad , a_n = \frac{4(-1)^n}{n^2}, \quad b_n = 0$$

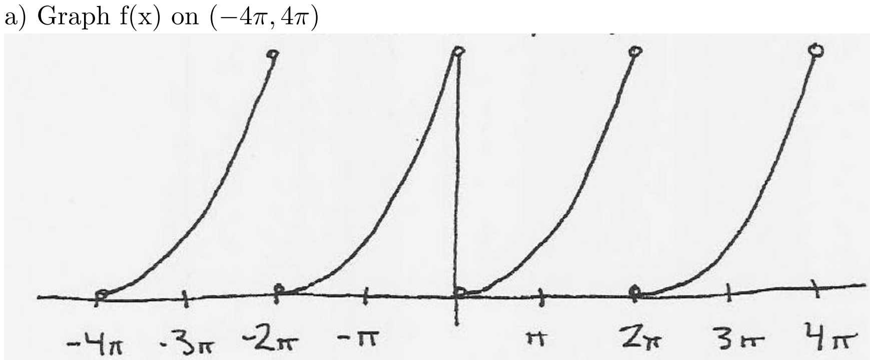
d) Graph the first five terms of the Fourier series.



3. Consider the Function

$$f(x) = x^{(}2) \quad x \in [0, 2\pi]$$

which is 2π periodic.
a) Graph f(x) on $(-4\pi, 4\pi)$



b) Is the function even, odd, or neither?
neither.
c) Determine the Fourier Coefficients a_0, a_n, b_n.

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} x^2 dx = \left. \frac{x^3}{6\pi} \right|_0^{2\pi} = \frac{{}^4\pi 2}{3}$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} x^2 cos(nx) dx =$$

$$= \frac{1}{\pi} \left[\frac{x^2}{n} sin(ns) + \frac{2x}{n^2} cos(nx) - \frac{2}{n^3} sin(nx) \right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{4\pi}{n^2}\right] = \frac{4}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x^2 sin(nx)dx$$

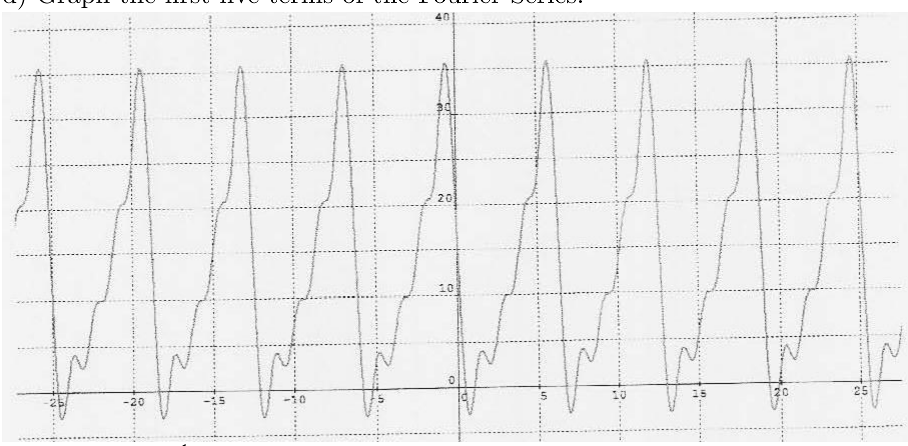
$$= \frac{1}{\pi}\left[\frac{-x^2}{n}cos(nx) + \frac{2x}{n^2}sin(nx) + \frac{2}{n^3}cos(nx)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{-4\pi^2}{n} + \frac{2}{n^3} - \frac{2}{n^3}\right] = \frac{-4\pi}{n}$$

$$f(x) = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\left[\frac{4}{n^2}cos(nx) - \frac{4\pi}{n}sin(nx)\right]$$

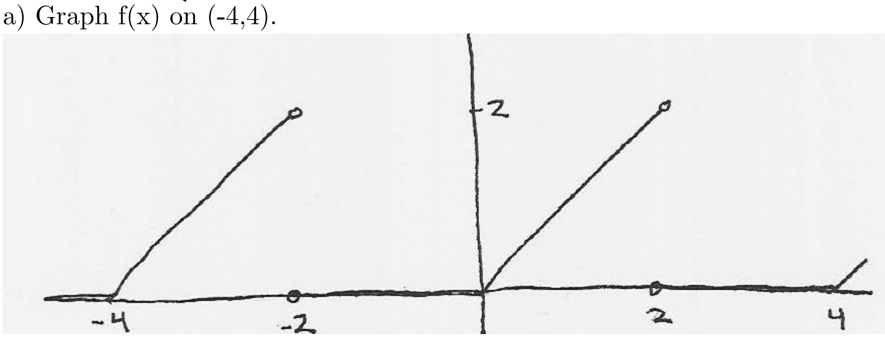
$$a_0 = \frac{4\pi^2}{3} \quad a_n = \frac{4}{n^2} \quad b_n = \frac{-4\pi}{n}$$

d) Graph the first five terms of the Fourier Series.



4. Let $f(x) = \begin{cases} 0 & -2 < x < 0 \\ x & 0 < x < 2 \end{cases}$ be a 4-periodic function.

a) Graph f(x) on (-4,4).



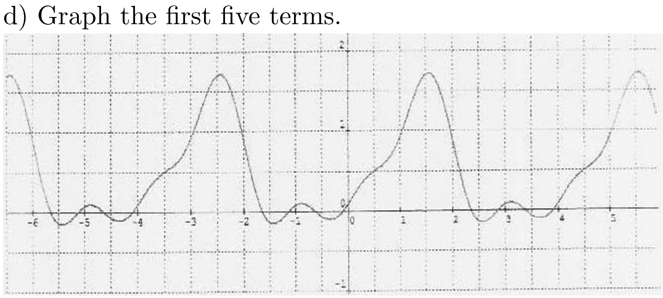
b) Is the function even, odd, or neither?
neither.

c) Determine the Fourier Coefficients a_0, a_n, b_n.

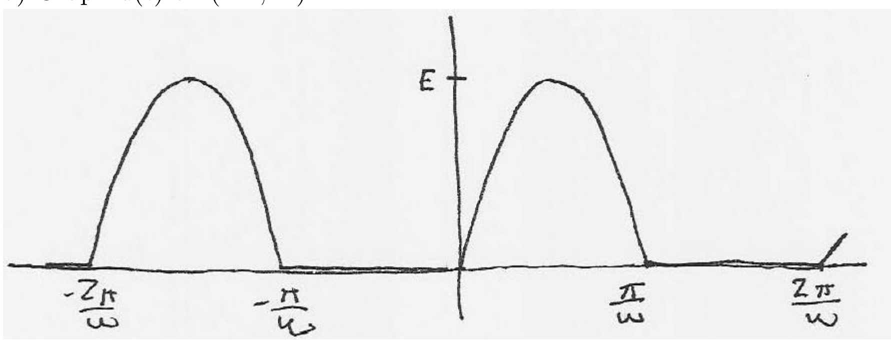
$$
\begin{aligned}
a_0 &= \frac{1}{2L}\int_{-1}^{1} f(x)dx = \frac{1}{4}\left[\int_{-2}^{0} 0dx + \int_{0}^{2} xdx\right] = \\
&= \frac{1}{8}x^2\Big|_0^2 = \frac{1}{2} \\
a_n &= \frac{1}{L}\int_{-L}^{L} f(x)cox(n\pi L)dx = \\
&= \frac{1}{2}\left[\int_{-2}^{0} 0\cdot cos(\frac{n\pi}{2}x)dx + \int_{0}^{2} x\cdot cos(\frac{n\pi}{2}x)dx\right] = \\
&= \frac{1}{2}\left[\frac{2x}{n\pi}sin(n\pi 2x) + \frac{4}{n^2\pi^2}cos(\frac{n\pi}{2}x)\right]_0^2 = \\
&= \frac{1}{2}\left[\frac{4}{n^2\pi^2}cos(n\pi) - \frac{4}{n^2\pi^2}\right] = \frac{2(-1)^n - 2}{n^2\pi^2} \\
b_n &= \frac{1}{L}\int_{-L}^{L} f(x)sin(\frac{n\pi}{L}x)dx = \\
&= \frac{1}{2}\left[\int_{-2}^{0} 0\cdot sin(\frac{n\pi}{2}x) + \int_{0}^{2} x\cdot sin(\frac{n\pi}{2}x)dx\right] = \\
&= \frac{1}{2}\left[\frac{-2x}{n\pi}cos(\frac{n\pi}{2}x) + \frac{4}{n^2\pi^2}sin(\frac{n\pi}{2}x)\right]_0^2 = \\
&= \frac{1}{2}\left[\frac{-4}{n\pi}(-1)^n\right] = \frac{2(-1)^{n+1}}{n\pi} \\
f(x) &= \frac{1}{2} + \sum_{n=1}^{\infty}\left[\frac{2(-1)^n - 2}{n^2\pi^2}cos(\frac{n\pi}{2}x) + \frac{2(-1)^{n+1}}{n\pi}sin(\frac{n\pi}{2}x)\right]
\end{aligned}
$$

$$a_0 = \frac{1}{2} \quad a_n = \frac{2(-1)^n - 2}{n^2\pi^2} \quad b_n = \frac{2(-1)^{n+1}}{n\pi}$$

d) Graph the first five terms.



5. Let $u(t) = \begin{cases} 0 & -L < t < 0 \\ E \cdot sin(wt) & 0 < t < L \end{cases}$ be a 2L-periodic function where E is the amplitude and w is the frequency.
We want L such that u(t) ranges over 1 full period of $E \cdot sin(wt)$.
The period of $E \cdot sin(wt) = \frac{2\pi}{w} = 2L$ thus, $L = \frac{\pi}{w}$.
a) Graph u(t) on (-2L,2L).



b) Is the function even, odd, or neither?
neither.
c) Determine the Fourier coefficients a_0, a_n, b_n.

$$
\begin{aligned}
a_0 &= \frac{1}{2L}\int_{-L}^{L} f(t)dt = \frac{w}{2\pi}\left[\int_{-\pi/w}^{0} 0 \cdot dt + \int_{0}^{\pi/w} E \cdot sin(wt)dt\right] = \\
&= \frac{-E}{2\pi}cos(wt)\Big|_0^{\pi/w} = \frac{-E}{2\pi}cos(\pi) + \frac{E}{2\pi}cos(0) = \\
&= \frac{E}{\pi} \\
a_n &= \frac{1}{L}\int_{-L}^{L} u(t)cos(\frac{n\pi}{L}t)dt = \\
&= \frac{w}{\pi}\left[\int_{-\pi/w}^{0} 0 \cdot cos(nwt)dt + \int_{0}^{\pi/w} E \cdot sin(wt)cos(nwt)dt\right] = \\
&= \frac{Ew}{2\pi}\left[\int_{0}^{\pi/w} sin(1+n)wt + sin(1-n)wt \cdot dt\right] = \\
&= \frac{Ew}{2\pi}\left[\frac{-cos(1+n)wt}{(1+n)w} - \frac{cos(1-n)wt}{(1-n)w}\right]_0^{\pi/w} = \\
&= \frac{E}{2\pi}\left[\frac{-(-1)^{1+n}}{1+n} - \frac{(-1)^{1-n}}{1-n} + \frac{1}{1+n} + \frac{1}{1-n}\right] = \\
&= \frac{E}{2\pi}\left[\frac{(-1)^n + 1}{1+n} + \frac{(-1)^n + 1}{1-n}\right] \quad n = 2, 3, 4 \ldots \\
& \text{Case } n = 1 \\
a_1 &= \frac{Ew}{\pi}\int_{0}^{\pi/w} sin(wt)cos(wt)dt = \frac{Ew}{2\pi}\int_{0}^{\pi/w} sin(2wt)dt =
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{Ew}{2\pi}\left[\frac{-cos(2wt)}{2w}\right]_0^{\pi/w} = \frac{E}{4\pi}\left[-cos(2\pi)+cos(0)\right] = \\
&= 0 \\
b_n &= \frac{1}{L}\int_{-L}^{L} u(t)sin(\frac{n\pi}{L}t)dt = \\
&= \frac{w}{\pi}\left[\int_{-\pi/w}^{0} 0\cdot sin(nwt)dt + \int_0^{\pi/w} E\cdot sin(wt)sint(nwt)dt\right] = \\
&= \frac{Ew}{2\pi}\left[\int_0^{\pi/w} cos(1-n)wt + cos(1+n)wt\cdot dt\right] = \\
&= Ew2\pi\left[\frac{sin(1-n)wt}{(1-n)w} + \frac{sin(1+n)wt}{(1+n)w}\right]_0^{\pi/w} = \\
&= 0 \quad n = 2,3,4\ldots \\
& \text{Case } n = 1 \\
b_1 &- \frac{Ew}{\pi}\int_0^{\pi/w} sin(wt)sin(wt)dt = \frac{Ew}{\pi}\int_0^{\pi/w} sin^2(wt)dt = \\
&= \frac{Ew}{2\pi}\int_0^{\pi/w} 1 - cos(2wt)dt = \\
&= \frac{Ew}{2\pi}\left[t - \frac{sin(2wt)}{2w}\right]_0^{\pi/w} = \frac{Ew}{2\pi}\left[\frac{\pi}{w}\right] = \frac{E}{2}
\end{aligned}
$$

$a_0 = \frac{E}{\pi}$

$a_1 = 0$

$a_n = \frac{E}{2\pi}\left[\frac{(-1)^n+1}{1+n} + \frac{(-1)^n+1}{1-n}\right] \quad n = 2,3,4\ldots$

$b_1 = \frac{E}{2}$

$b_n = 0 \quad n = 2,3,4\ldots$

$$
u(t) = \frac{E}{\pi} + \frac{E}{2}sin(wt) + \sum_{n=2}^{\infty}\frac{E}{2\pi}\left[\frac{(-1)^n+1}{1+n} + \frac{(-1)^n+1}{1-n}\right]cos(nwt)
$$

d) Graph the first five terms of the Fourier Series.