MATH 348 - Advanced Engineering Mathematics July 15, 2009

Homework 9, Summer 2009 **Due**: July 30, 2009

REVIEW OF ORDINARY DIFFERENTIAL EQUATIONS - INTRODUCTION TO PARTIAL DIFFERENTIAL EQUATIONS

1. Given,
$$ay'' + by' + cy = f(x). \tag{1}$$
   (a) Find all solutions to the homogeneous version of (1). List this information in a table based on the value of $D = b^2 - 4ac$. [1]

   (b) Fill in the following table, which outlines the choices you would make for the particular solution for various choices of $f(x)$:[2]

$f(x)$	$y_p(x)$
x^4	
$\cos(\beta x)$	
$e^{\alpha x} + \sin(x) + x$	
$e^{\alpha x}\sin(\beta x)$	
$x^2 e^{\alpha x}\cos(\beta x)$	

   (c) Suppose that $a = 1$, $b = 0$, $c = 9$ and $f(x) = \cos(3x)$. Find the general solution to (1).[3]

2. Consider the boundary value problem:
$$\begin{aligned} y'' + \lambda y &= 0 \quad (2)\\ y'(0) = y'(L) &= 0, \quad L \in \mathbb{R}^+, \quad (3)\end{aligned}$$
   (a) Assuming that $\lambda \in \mathbb{R}$, find the general solution to (1).

   (b) Apply the boundary conditions (2) to the solutions found in (a) and calculate the possible values of λ such that these solutions also satisfy the boundary conditions.

   Hint: For problem (a) use the assumption that $y(x) = e^{rx}$, $r \in \mathbb{R}$ and find solutions for the three cases $\lambda < 0$, $\lambda > 0$, $\lambda = 0$. Concerning (b), this time, two of the three solution types will non-trivially satisfy (1)-(2).

[1] First assume that $y(x) = e^{rx}$ to find the characteristic equation $ar^2 + br + c = 0$. After this find the roots of the characteristic equation and thus the general solutions to (1). Note that these solutions will depend on the discriminant of the characteristic equation.

[2] Do not solve for the undetermined coefficients. Also, you may assume that a, b, c are such that no part of the homogeneous solution matches your particular solutions.

[3] Keep in mind that this is a resonant case and the choice for particular solution will need a coefficient of x. Don't forget to use the product rule!

3. Consider the ordinary differential equation:

$$y'' - y = 0 \tag{4}$$

We know that the general solution to this equation is $y(x) = c_1 e^x + c_2 e^{-x}$. It is common to write the solutions to (4) in terms of the hyperbolic trigonometric functions, $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$, $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$. [4]

(a) Show that $y(x) = b_1 \sinh(x) + b_2 \cosh(x)$ is a solution to the differential equation (3).

(b) Show that if $c_1 = \dfrac{b_1 + b_2}{2}$ and $c_2 = \dfrac{b_1 - b_2}{2}$ then $y(x) = c_1 e^x + c_2 e^{-x} = b_1 \cosh(x) + b_2 \sinh(x)$.

(c) Assume that $y(x) = \displaystyle\sum_{n=0}^{\infty} a_n x^n$ to find the general solution of (3) in terms of the hyperbolic sine and cosine functions. [5]

Hint: For (c) you should find the recurrence relation $a_{n+2} = \dfrac{a_n}{(n+2)(n+1)}$. Evaluating this recurrence relation for even-n will give the Taylor coefficients for hyperbolic cosine the odd-n will give Taylor coefficients for hyperbolic sine found in (4).

4. In class we will study the heat and wave equation in one spatial dimension. Though these problems are considered rather plain they exist as the basic components for the study of flows and vibrations. Consider the following reference material,

- `http://en.wikipedia.org/wiki/Heat_equation`
- `http://en.wikipedia.org/wiki/Diffusion_equation`
- `http://en.wikipedia.org/wiki/Wave_equation`
- `http://en.wikipedia.org/wiki/Dispersion_%28water_waves%29`

and respond to the following questions.

(a) For the heat equation u represents the temperature of the medium as a function of space and time. It can be shown that u obeys the maximum principle. Explain the physical interpretation of this principle.

(b) Explain the relationship between the heat equation and the diffusion equation as given by the websites. What is the fundamental principle used to derive the diffusion equation?

(c) What physical phenomenon modeled by the linear wave equation? What about the nonlinear wave equation?

(d) Define dispersion and give a physical example of dispersion in travailing waves.

5. Show that the following functions are solutions to their corresponding PDE's.

(a) $u(x,t) = f(x - ct) + g(x + ct)$ for the 1-D wave equation.

(b) $u(x,t) = e^{-4\omega^2 t}\sin(\omega x)$ where $c = 2$ for the 1-D heat equation.

(c) $u(x,y) = x^4 + y^4$ where $f(x,y) = 12(x^2 + y^2)$ for the 2-D Poisson equation.

(d) $u(x,y,z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$ for the 3-D Laplace equation.

Note: A listing of the PDE's can be found on page 536 of Kreyszig.

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[4] After you finish this problem you may want to look back at the previous problem and see if you can reformulate its solutions in terms of hyperbolic trigonometric functions. This will cause the $\lambda > 0$ case to look similar to the $\lambda < 0$ case, which is a primary reason for using these functions.

[5] The hyperbolic sine and cosine have the following Taylor's series representations centred about $x = 0$,

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \qquad \sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}. \tag{5}$$

It is worth noting that these are basically the same Taylor series as cosine/sine with the exception that the signs of the terms do not alternate. From this we can gather a final connection for wrapping all of these functions together. If you have the Taylor series for the exponential function and extract the even terms from it then you have the hyperbolic cosine function. Taking the hyperbolic cosine function and alternating the sign of its terms gives you the cosine function. Extracting the odd terms from the exponential function gives the same statements for the hyperbolic sine and sine functions. The reason these functions are connected via the imaginary number system is because when i is raised to integer powers it will produce these exact sign alternations. So, if you remember $e^x = \sum_{n=0}^{\infty} x^n/n!$ and $i = \sqrt{-1}$ then the rest (hyperbolic and non-hyperbolic trigonometric functions) follows!