EM waves and interference

Review of EM wave equation and plane waves

Energy and intensity in EM waves

Interference

Maxwell's Equations to wave eqn

- The induced polarization, **P**, contains the effect of the medium:

$$\vec{\nabla} \cdot \mathbf{E} = 0 \qquad \vec{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\vec{\nabla} \cdot \mathbf{B} = 0 \qquad \vec{\nabla} \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} + \mu_0 \frac{\partial \mathbf{P}}{\partial t}$$

Take the curl:

$$\vec{\nabla} \times \left(\vec{\nabla} \times \mathbf{E}\right) = -\frac{\partial}{\partial t}\vec{\nabla} \times \mathbf{B} = -\frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} + \mu_0 \frac{\partial \mathbf{P}}{\partial t}\right)$$

Use the vector ID:

$$\mathbf{A} \times \left(\mathbf{B} \times \mathbf{C}\right) = \mathbf{B}\left(\mathbf{A} \cdot \mathbf{C}\right) - \mathbf{C}\left(\mathbf{A} \cdot \mathbf{B}\right)$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \mathbf{E}\right) = \vec{\nabla}\left(\vec{\nabla} \cdot \mathbf{E}\right) - \left(\vec{\nabla} \cdot \vec{\nabla}\right)\mathbf{E} = -\vec{\nabla}^2 \mathbf{E}$$

$$\boxed{\vec{\nabla}^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2}} \qquad \text{"Inhomogeneous Wave Equation"}$$

Wave equation in a medium

- The induced polarization, **P**, contains the effect of the medium:

$$\vec{\nabla}^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2}$$

- Sinusoidal waves of all frequencies are solutions to the wave equation
- The polarization (**P**) is sometimes a driving term for the waves (esp in nonlinear optics). In this case the polarization determines which frequencies will occur.
- For linear response, **P** will oscillate at the same frequency as the input.

$$\mathbf{P}(\mathbf{E}) = \varepsilon_0 \chi \mathbf{E}$$

- In nonlinear optics, the induced polarization is more complicated:

$$\mathbf{P}(\mathbf{E}) = \varepsilon_0 \left(\chi^{(1)} \mathbf{E} + \chi^{(2)} \mathbf{E}^2 + \chi^{(3)} \mathbf{E}^3 + ... \right)$$

- The extra nonlinear terms can lead to new frequencies.

Solving the wave equation: linear induced polarization

For low irradiances, the polarization is proportional to the incident field:

$$\mathbf{P}(\mathbf{E}) = \varepsilon_0 \chi \mathbf{E}$$

The displacement vector combines the effect of E and P

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon_0 (1 + \chi) \mathbf{E} = \varepsilon \mathbf{E} = n^2 \mathbf{E}$$

In this simple (and most common) case, the wave equation becomes:

$$\vec{\nabla}^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{1}{c^2} \chi \frac{\partial^2 \mathbf{E}}{\partial t^2} \qquad \rightarrow \vec{\nabla}^2 \mathbf{E} - \frac{n^2}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

Using: $\varepsilon_0 \mu_0 = 1/c^2$ $\qquad\qquad \varepsilon_0(1+\chi) = \varepsilon = n^2$

D does not always point in the same direction as **E**:

- Birefringence: n is a function of linear polarization direction
- Optically activity: n is a function of circular polarization state
- In this case, χ and ε are tensors operating on **E** to change its direction

Vector wave equation

- The EM wave equation in vector form

$$\vec{\nabla}^2 \mathbf{E} - \frac{n^2}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

- **E** is a vector that depends on **r** and t
- Wave equation is actually a set of 3 equations:

$$\vec{\nabla}^2 E_x(\mathbf{r},t) - \frac{n^2}{c^2} \frac{\partial^2}{\partial t^2} E_x(\mathbf{r},t) = 0$$

$$\vec{\nabla}^2 E_y(\mathbf{r},t) - \frac{n^2}{c^2} \frac{\partial^2}{\partial t^2} E_y(\mathbf{r},t) = 0$$

$$\vec{\nabla}^2 E_z(\mathbf{r},t) - \frac{n^2}{c^2} \frac{\partial^2}{\partial t^2} E_z(\mathbf{r},t) = 0$$

Plane wave solutions for the wave equation

If we assume the solution has no dependence on x or y:

$$\vec{\nabla}^2 \mathbf{E}(z,t) = \frac{\partial^2}{\partial x^2}\mathbf{E}(z,t) + \frac{\partial^2}{\partial y^2}\mathbf{E}(z,t) + \frac{\partial^2}{\partial z^2}\mathbf{E}(z,t) = \frac{\partial^2}{\partial z^2}\mathbf{E}(z,t)$$

$$\rightarrow \frac{\partial^2 \mathbf{E}}{\partial z^2} - \frac{n^2}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

The solutions are oscillating functions, for example

$$\mathbf{E}(z,t) = \hat{\mathbf{x}} E_x \cos(k_z z - \omega t)$$

Where $\omega = k c, \quad k = 2\pi n / \lambda, \quad v_{ph} = c / n$

This is a linearly polarized wave.

Complex notation for waves

- Write cosine in terms of exponential

$$\mathbf{E}(z,t) = \hat{\mathbf{x}} E_x \cos(kz - \omega t + \phi) = \hat{\mathbf{x}} E_x \frac{1}{2}\left(e^{i(kz-\omega t+\phi)} + e^{-i(kz-\omega t+\phi)}\right)$$

- Note E-field is a *real* quantity.
- It is convenient to work with just one part
  - We will use $\quad E_0 e^{+i(kz-\omega t)}$
  - Svelto: $\quad E_0 e^{-i(kz-\omega t)}$
- Then take the real part.
  - No factor of 2
  - In *nonlinear* optics, we have to explicitly include conjugate term

Wave energy and intensity

- Both E and H fields have a corresponding energy density (J/m^3)

  – For static fields (e.g. in capacitors) the energy density can be calculated through the work done to set up the field

  $$\rho = \tfrac{1}{2}\varepsilon E^2 + \tfrac{1}{2}\mu H^2$$

  – Some work is required to polarize the medium

  – Energy is contained in both fields, but H field can be calculated from E field

+Q   −Q

Dielectric
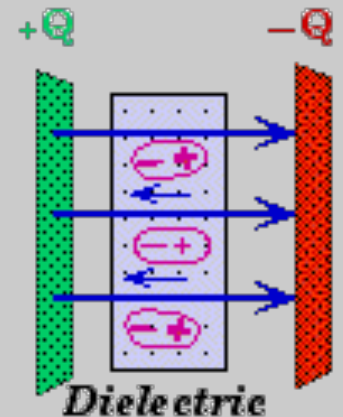
Calculating H from E in a plane wave

- Assume a non-magnetic medium

$$\mathbf{E}(z,t) = \hat{\mathbf{x}} E_x \cos(kz - \omega t)$$

$$\vec{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}$$

 – Can see **H** is perpendicular to **E**

$$-\mu_0 \frac{\partial \mathbf{H}}{\partial t} = \vec{\nabla} \times \mathbf{E} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_x & \partial_y & \partial_z \\ E_x & 0 & 0 \end{vmatrix} = \hat{\mathbf{y}} \partial_z E_x = -\hat{\mathbf{y}} k E_x \sin(kz - \omega t)$$

 – Integrate on time to get **H**-field:

$$\mathbf{H} = \hat{\mathbf{y}} \int \frac{k E_x}{\mu_0} \sin(kz - \omega t) \, dt = \hat{\mathbf{y}} \frac{k E_x}{\mu_0} \left(\frac{-\cos(kz - \omega t)}{-\omega} \right) = \hat{\mathbf{y}} \frac{k E_x}{\omega \mu_0} \cos(kz - \omega t)$$

H field from E field

- H field for a propagating wave is *in phase* with E-field

$$\mathbf{E}(z,t) = \hat{\mathbf{x}} E_x \cos(k z - \omega t)$$

$$\mathbf{H}(z,t) = \hat{\mathbf{y}} \frac{k E_x}{\omega \mu_0} \cos(k z - \omega t)$$

Electromagnetic Wave
Magnetic Field (B)
Electric Field (E)
Wavelength (λ)
Propagation Direction

Figure 1
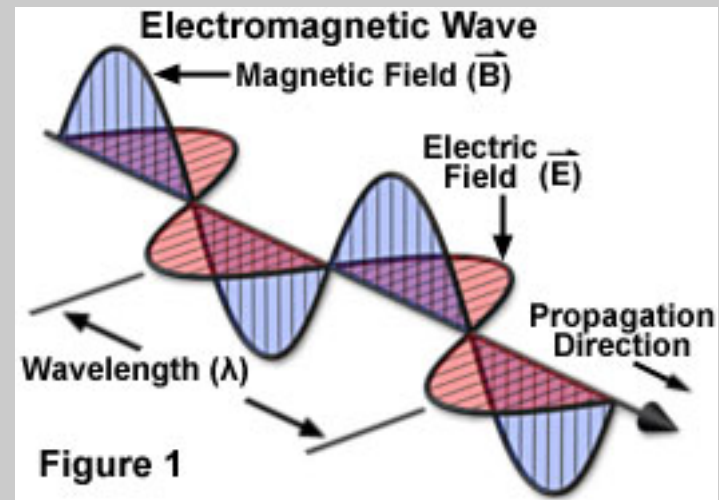
- Amplitudes are not independent

$$H_y = \frac{k}{\omega \mu_0} E_x \qquad k = n \frac{\omega}{c} \qquad c^2 = \frac{1}{\mu_0 \varepsilon_0} \rightarrow \frac{1}{\mu_0 c} = \varepsilon_0 c$$

$$H_y = \frac{n}{c \mu_0} E_x = n \varepsilon_0 c E_x$$

Energy density in a traveling EM wave

- Back to energy density, non-magnetic

$$\rho = \tfrac{1}{2}\varepsilon E^2 + \tfrac{1}{2}\mu_0 H^2$$

$$H = n\varepsilon_0 cE$$

$$\varepsilon = \varepsilon_0 n^2$$

$$\rho = \tfrac{1}{2}\varepsilon_0 n^2 E^2 + \tfrac{1}{2}\mu_0 n^2 \varepsilon_0^2 c^2 E^2$$

$$\mu_0 \varepsilon_0 c^2 = 1$$

$$\rho = \tfrac{1}{2}\varepsilon_0 n^2 E^2 + \tfrac{1}{2}\varepsilon_0 n^2 E^2 = \varepsilon_0 n^2 E_0^2 \cos^2\left(kz - \omega t\right)$$

Equal energy in both components of wave

Cycle-averaged energy density

- Optical oscillations are faster than detectors
- Average over one cycle:

$$\langle \rho \rangle = \varepsilon_0 n^2 {E_0}^2 \frac{1}{T} \int_0^T \cos^2\left(k_z z - \omega t\right) dt$$

  – Graphically, we can see this should = ½

k z = 0

k z = π/4
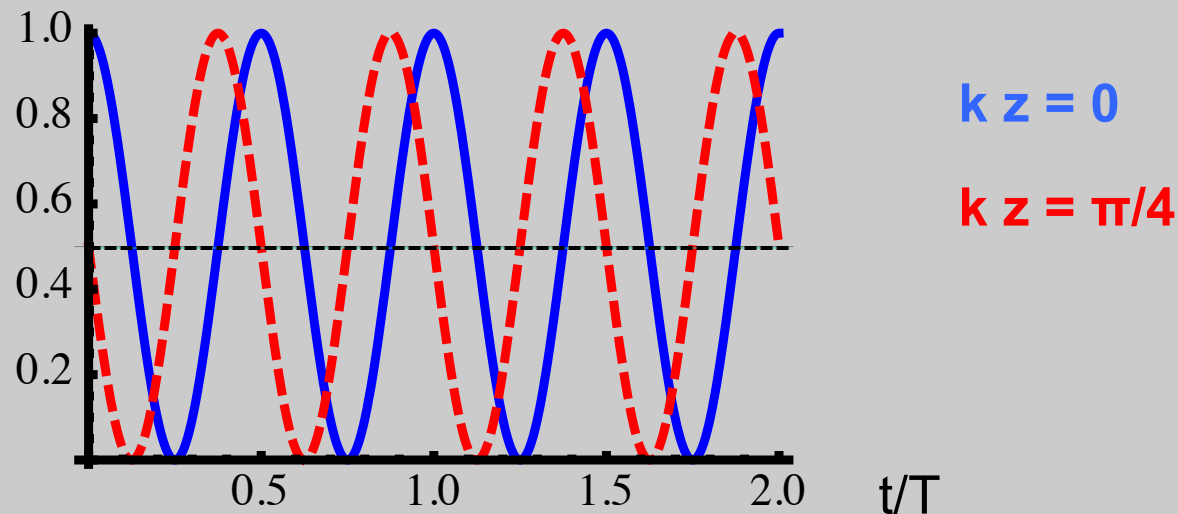
  – Regardless of position z

$$\boxed{\langle \rho \rangle = \frac{1}{2} \varepsilon_0 n^2 {E_0}^2}$$

Intensity and the Poynting vector

- Intensity is an energy flux (J/s/cm^2)
- In EM the Poynting vector give energy flux

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}$$

  – For our plane wave,

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = E_0 \cos\left(k_z z - \omega t\right) n\varepsilon_0 c E_0 \cos\left(k_z z - \omega t\right) \hat{\mathbf{x}} \times \hat{\mathbf{y}}$$

$$\mathbf{S} = n\varepsilon_0 c E_0^2 \cos^2\left(k_z z - \omega t\right) \hat{\mathbf{z}}$$

  – **S** is along **k**

- Time average: $\mathbf{S} = \frac{1}{2} n \varepsilon_0 c E_0^2 \hat{\mathbf{z}}$

- *Intensity* is the magnitude of **S**

$$I = \frac{1}{2} n\varepsilon_0 c E_0^2 = \frac{c}{n}\rho = V_{phase} \cdot \rho$$

Photon flux: $F = \dfrac{I}{h\nu}$

General 3D plane wave solution

- Assume separable function

$$\mathbf{E}(x,y,z,t) \sim f_1(x) f_2(y) f_3(z) g(t)$$

$$\vec{\nabla}^2 \mathbf{E}(z,t) = \frac{\partial^2}{\partial x^2}\mathbf{E}(z,t) + \frac{\partial^2}{\partial y^2}\mathbf{E}(z,t) + \frac{\partial^2}{\partial z^2}\mathbf{E}(z,t) = \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E}(z,t)$$

- Solution takes the form:

$$\mathbf{E}(x,y,z,t) = \mathbf{E_0}e^{ik_x x}e^{ik_y y}e^{ik_z z}e^{-i\omega t} = \mathbf{E_0}e^{i(k_x x + k_y y + k_z z)}e^{-i\omega t}$$

$$\mathbf{E}(x,y,z,t) = \mathbf{E_0}e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$

  – Now k-vector can point in arbitrary direction

- With this solution in W.E.:

$$\boxed{n^2\frac{\omega^2}{c^2} = k_x^2 + k_y^2 + k_z^2 = \mathbf{k}\cdot\mathbf{k}}$$

Valid even in waveguides
and resonators

Grad and curl of 3D plane waves

- Simple trick:

  $$\nabla \cdot \mathbf{E} = \partial_x E_x + \partial_y E_y + \partial_z E_z$$

  – For a plane wave,

  $$\nabla \cdot \mathbf{E} = i\left(k_x E_x + k_y E_y + k_z E_z\right) = i\left(\mathbf{k} \cdot \mathbf{E}\right)$$

  – Similarly,

  $$\nabla \times \mathbf{E} = i\left(\mathbf{k} \times \mathbf{E}\right)$$

- Consequence: since $\quad \nabla \cdot \mathbf{E} = 0, \quad \mathbf{k} \perp \mathbf{E}$

  – For a given **k** direction, **E** lies in a plane
  – E.g. **x** and **y** linear polarization for a wave propagating in **z** direction

Writing electric field expressions

- Write an expression for a complex E-field as shown:
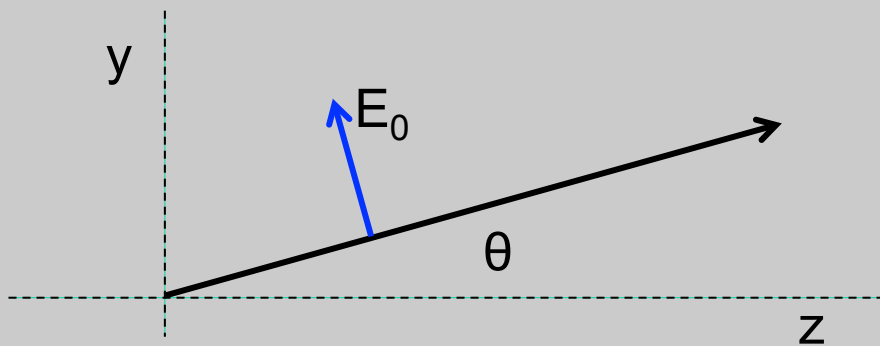
$$\mathbf{E}(y,z,t) = E_0 \left[\hat{\mathbf{y}} \cos\theta - \hat{\mathbf{z}} \sin\theta \right] e^{i(ky\sin\theta + kz\cos\theta - \omega t)}$$

Interference: ray and wave pictures

- Interference results from the sum of two waves with different *phase*:

$$E_{tot}(\Delta\phi) = E_1 e^{ikz} + E_2 e^{ikz+\Delta\phi}$$

- We measure intensity, which leads to interference

$$I_{tot}(\Delta\phi) \propto \left| E_1 e^{ikz} + E_2 e^{ikz+\Delta\phi} \right|^2 = \left| E_1 + E_2 e^{i\Delta\phi} \right|^2$$

$$= I_1 + I_2 + \sqrt{I_1 I_2}\, e^{i\Delta\phi} + \sqrt{I_1 I_2}\, e^{-i\Delta\phi}$$

$$= I_1 + I_2 + 2\sqrt{I_1 I_2}\, \cos(\Delta\phi)$$

  – For the case where $I_1 = I_2$,

$$I_{tot}(\Delta\phi) = 2I(1 + \cos(\Delta\phi)) = 4I \cos^2(\Delta\phi / 2)$$

- How to generate, calculate phase difference?

The Fizeau Wedge Interferometer

The Fizeau wedge yields a complex pattern of variable-width fringes, but it can be used to measure the wavelength of a laser beam.
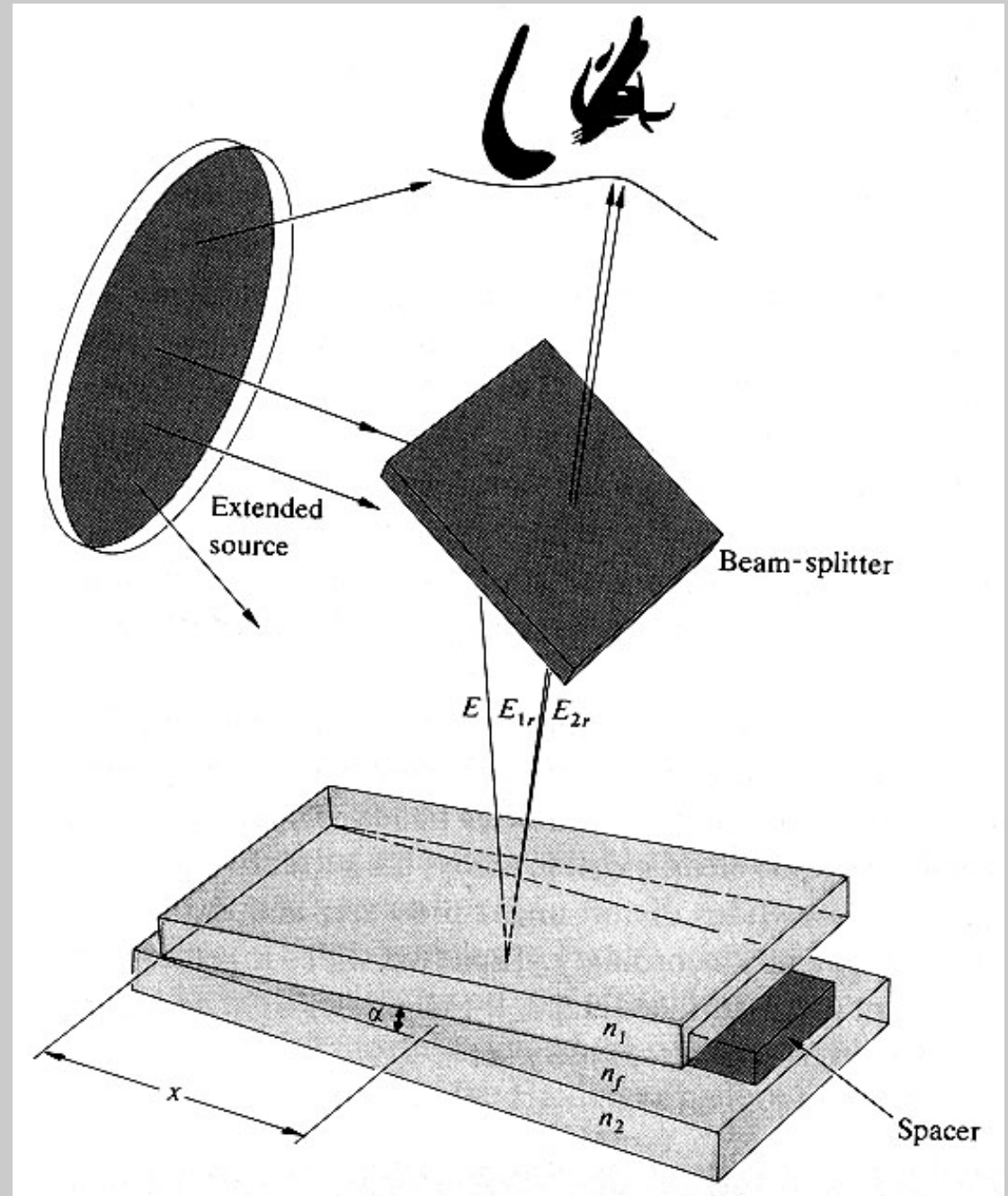
Fizeau wedge calculation

- Interference between reflections from internal surfaces

α

L
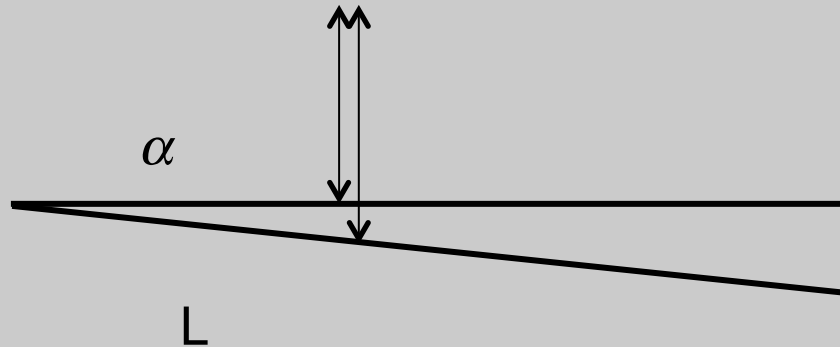
  – Angle is very small, neglect change in direction
  – Path difference: $\Delta l = 2L\sin\alpha \approx 2L\alpha$
  – Phase difference: $\Delta\phi = \dfrac{\omega}{c} n \Delta l \approx 2\pi \dfrac{2L}{\lambda}\alpha$ n = 1

  – Interference: $I_{tot}(\Delta\phi) = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos(\Delta\phi)$
  – One fringe from one max to the next, so maxima are at $\Delta\phi = 2\pi m$

- In this interferometer, minimum path = 0, we can measure absolute wavelength: $\Delta\phi = 2\pi m = 2\pi\dfrac{2L}{\lambda}\alpha \rightarrow \lambda = \dfrac{2L}{m}\alpha$

Newton's Rings
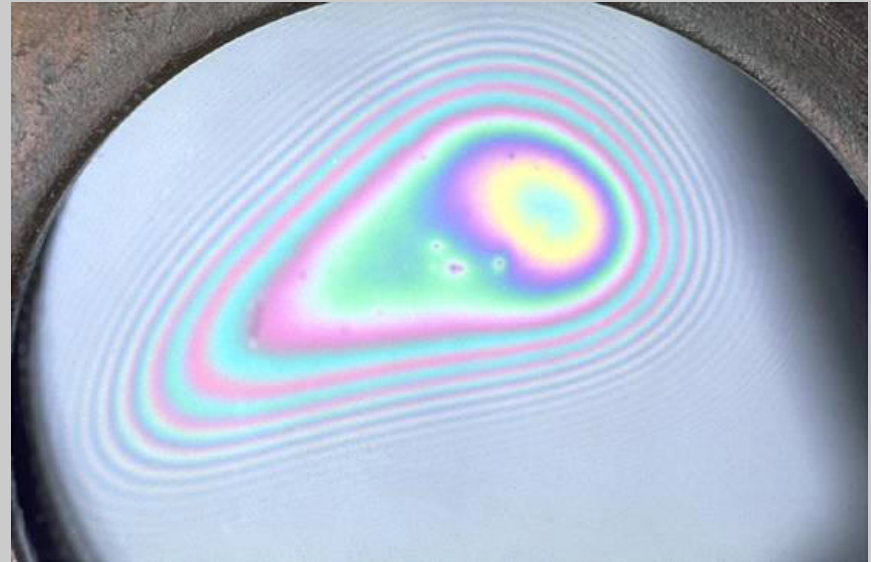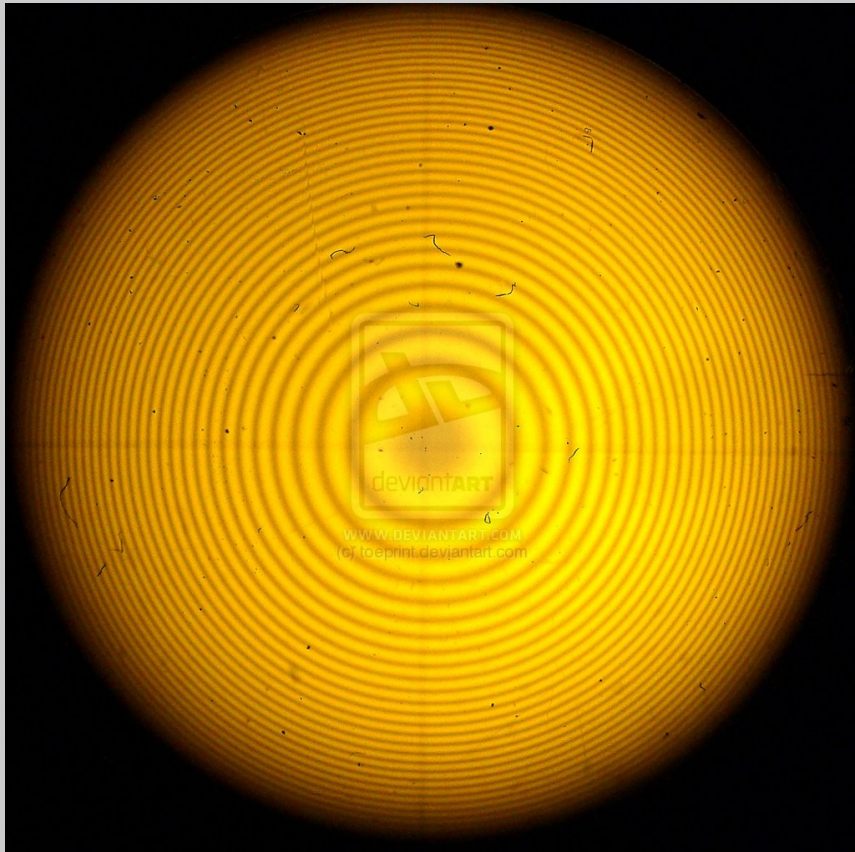
Newton's Rings

Get constructive interference when an integral number of half wavelengths occur between the two surfaces (that is, when an integral number of full wavelengths occur between the path of the transmitted beam and the twice reflected beam).

Twice reflected beam

Transmitted beam

Curved surface

Flat surface

Incident beam
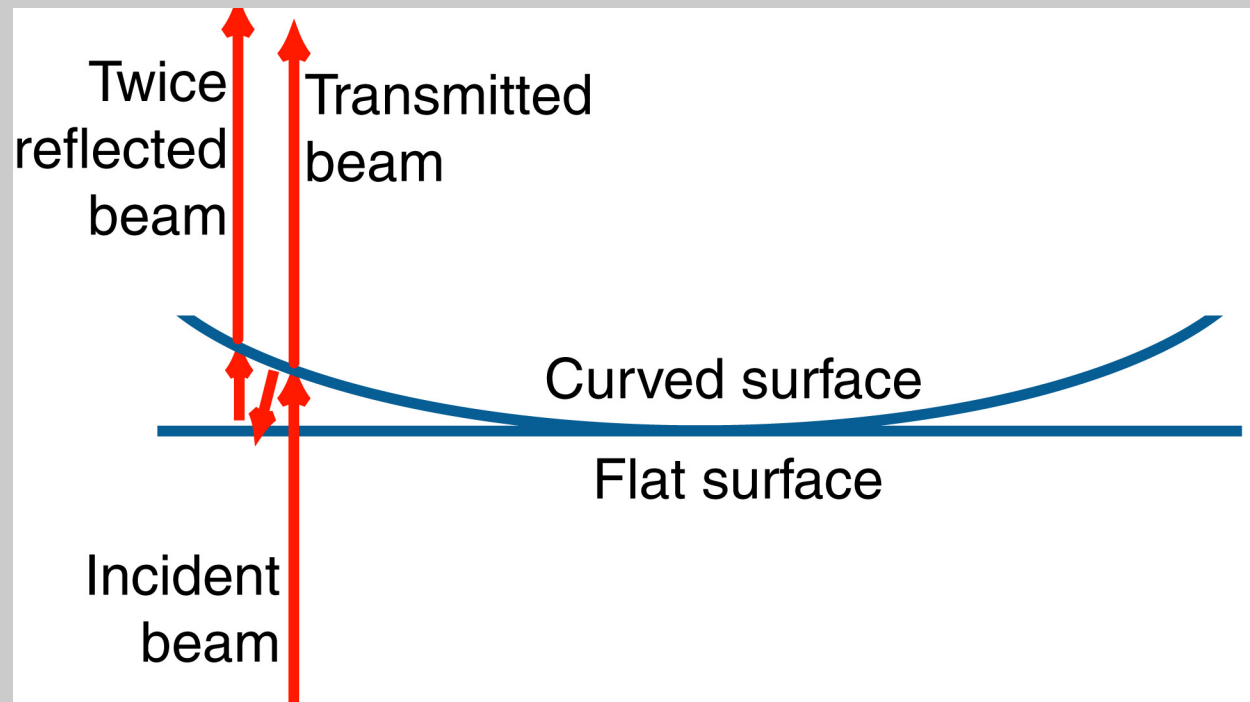
This effect also causes the colors in bubbles and oil films on puddles.

Curved wavefronts

- Rays are directed normal to surfaces of constant phase
  - These surfaces are the wavefronts
  - Radius of curvature is approximately at the focal point
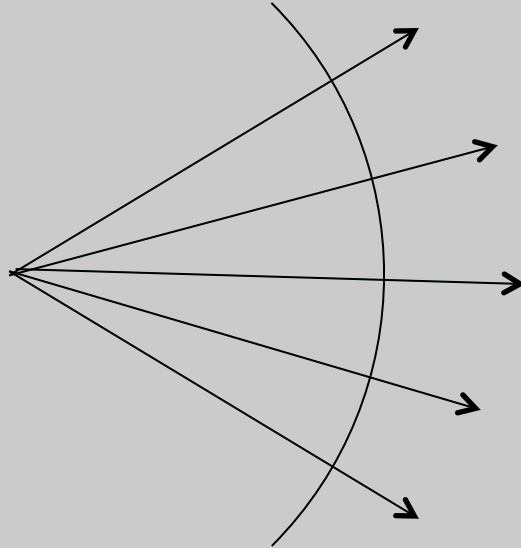
- Spherical waves are solutions to the wave equation (away from r = 0)

$$\nabla^2 E + \frac{n^2 \omega^2}{c^2} E = 0$$

$$E \propto \frac{1}{r} e^{i(\pm kr - \omega t)}$$

Scalar r
+ outward
- inward

$$I \propto \frac{1}{r^2}$$

Paraxial approximations

- For **rays**, paraxial = small angle to optical axis
  - Ray slope: $\tan\theta \approx \theta$

- For **spherical waves** where power is directed forward:

$$e^{ikr} = \exp\left[ik\sqrt{x^2 + y^2 + z^2}\right]$$

$$k\sqrt{x^2 + y^2 + z^2} = k\,z\sqrt{1 + \frac{x^2 + y^2}{z^2}} \approx k\,z\left(1 + \frac{x^2 + y^2}{2z^2}\right)$$

Expanding to
1st order

$$e^{i(kr - \omega t)} \rightarrow e^{ikz}\exp\left[i\left(k\frac{x^2 + y^2}{2z} - \omega t\right)\right]$$

z is radius of curvature = *R*

Wavefront = surface of constant phase
For x, y >0, t must increase.
Wave is diverging:
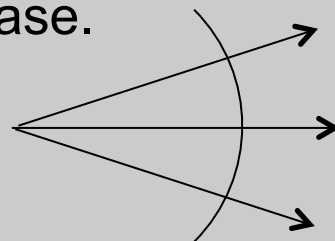
$$\phi = 0 \rightarrow k\frac{x^2 + y^2}{2R} = \omega t$$

Newton's rings: interfere plane and spherical waves

- Add two fields:

$$E(r) = E_0 + E_0 e^{i\frac{kr^2}{2R}}$$

Twice reflected beam | Transmitted beam
Curved surface
Flat surface
Incident beam

  – Assume equal amplitude
  – For Newton's rings, 2x phase shift

- Calculate intensity:

$$I(r) \propto \left| E_0 + E_0 e^{i\frac{kr^2}{2R}} \right|^2 = 2E_0^2 + 2E_0^2 \cos\left(\frac{kr^2}{2R}\right)$$
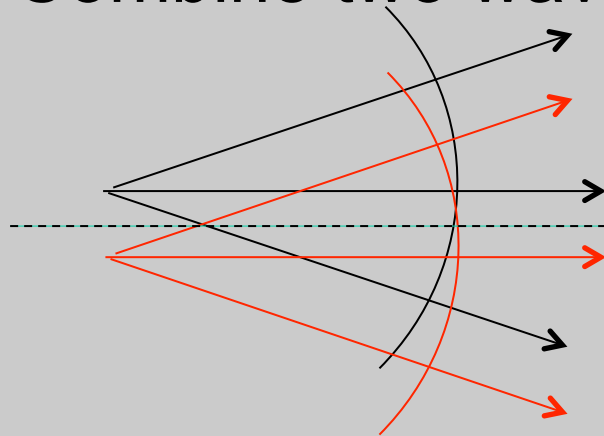
  – Local k increases with r

$$\cos\left(\frac{kr}{2R}r\right) = \cos\left(k_{local} r\right)$$
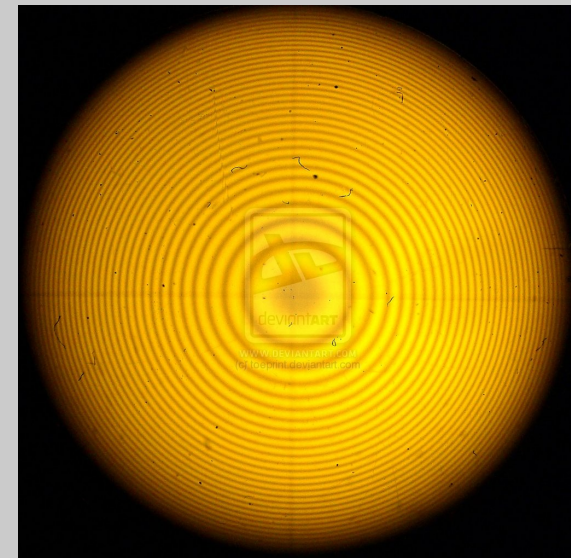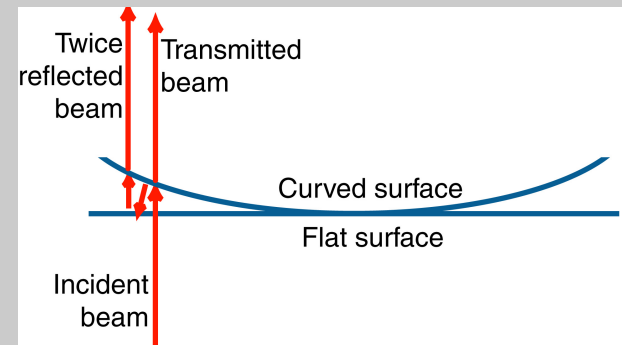
Shearing interferometer

- Combine two waves with a lateral offset ("shear")

$$\downarrow \; \mathsf{x_s} \quad E_{tot}(x) = E_0 \exp\left[i\frac{k}{2R}\left((x - x_s)^2 + y^2 \right) \right]$$

$$+ E_0 \exp\left[i\frac{k}{2R}\left((x + x_s)^2 + y^2 \right) \right]$$

$$I_{tot}(x) = I_0 \left(2 + \exp\left[i\left(\frac{k(x - x_s)^2}{2R} - \frac{k(x + x_s)^2}{2R} \right) \right] + c.c. \right)$$

$$(x - x_s)^2 - (x + x_s)^2 = x^2 - 2xx_s + x_s^2 - \left(x^2 + 2xx_s + x_s^2 \right) = 4xx_s$$

$$I_{tot}(x) = 2I_0 \left(1 + \cos\left[\frac{2kx_s}{R} x \right] \right)$$

Fringes are straight, equally spaced
Combine with constant tilt in y direction:
leads to fringe *rotation* with divergence

Tilted window: ray propagation

- Calculate phase shift caused by the insertion of the window into an interferometer.

- Ray optics:
  - Add up optical path for each segment
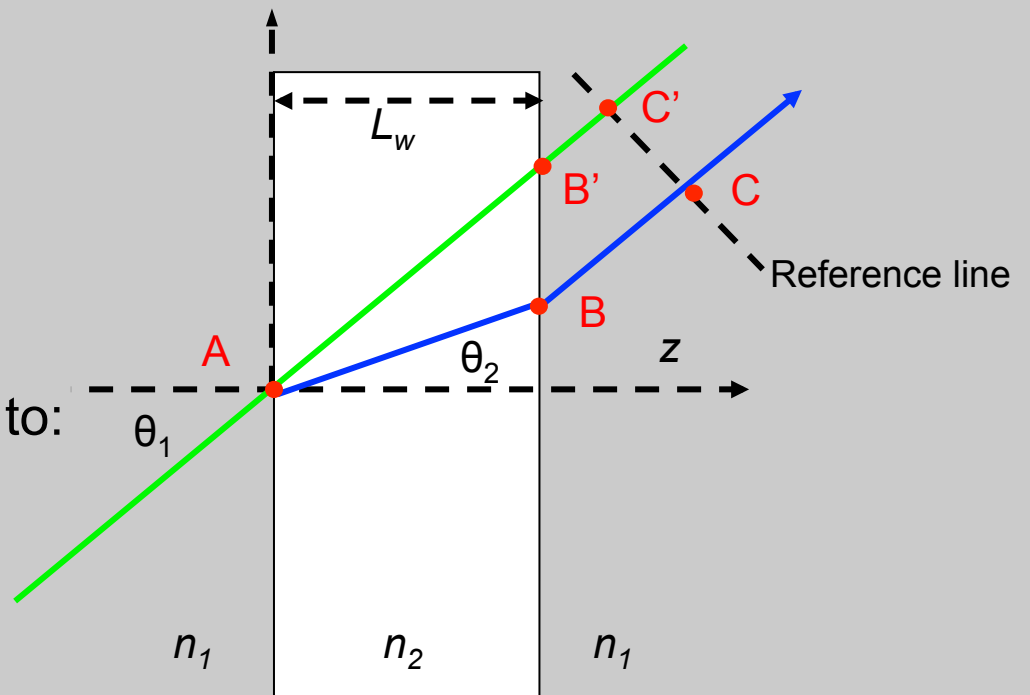 - Subtract optical path w/o window

$$\Delta d = n L_{AB} + L_{BC} - L_{AB'} - L_{B'C'}$$

$$L_{AB} = \frac{L_w}{\cos\theta_2} \qquad L_{AB'} = \frac{L_w}{\cos\theta_1}$$

$$L_{BC} = L_{B'C'} + L_{BB'}\sin\theta_1$$

  - Use Snell's Law to reduce to:

$$\Delta d = n\,L_w\cos\theta_2 - L_w\cos\theta_1$$

L_w

C'

B'

C

Reference line

B

z

A

θ_2

θ_1

n_1 n_2 n_1

Tilted window: wave propagation

- Write expression for tilted plane wave

$$E(x,z) = E_0 \exp\left[i\left(k_x x + k_z z\right)\right] = E_0 \exp\left[i\frac{\omega}{c}n\left(x\sin\theta_2 + z\cos\theta_2\right)\right]$$

- Snell's Law: phase across surfaces is conserved

$$k_x x = \frac{\omega}{c}n\sin\theta \quad \textbf{is constant}$$

$$\Delta\phi = \left(k_2\cos\theta_2\right)L_w - \left(k_1\cos\theta_1\right)L_w$$

- This approach can be used to calculate phase of prism pairs and grating pairs
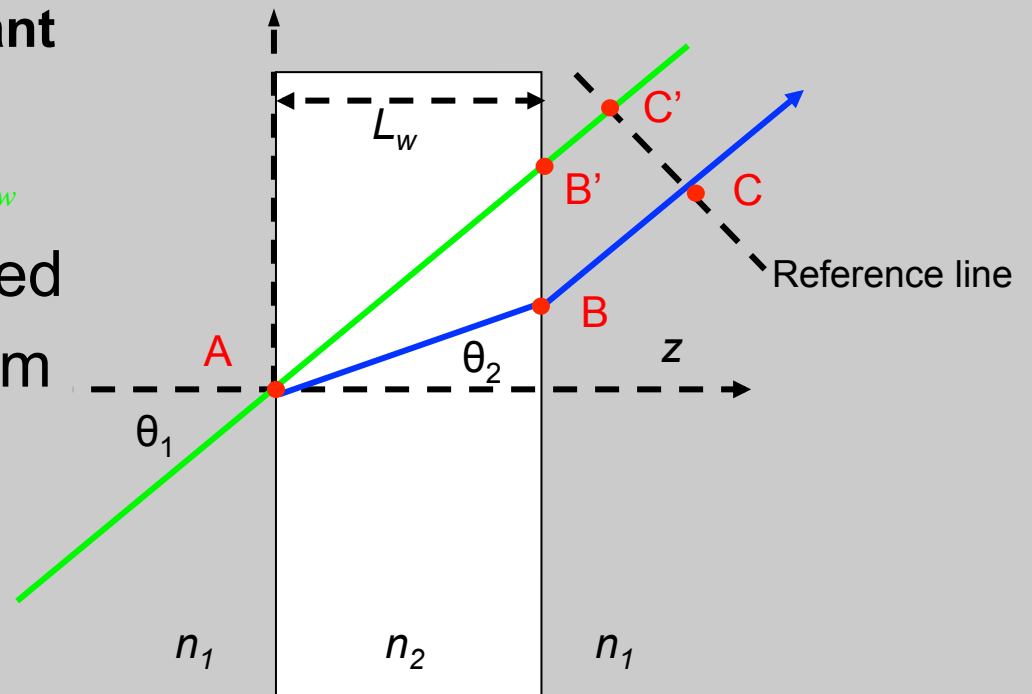
Multiple-beam interference:
The Fabry-Perot Interferometer or Etalon

A Fabry-Perot interferometer is a pair of **parallel** surfaces that reflect beams back and forth. An etalon is a type of Fabry-Perot etalon, and is a piece of glass with parallel sides.

The transmitted wave is an infinite series of multiply reflected beams.
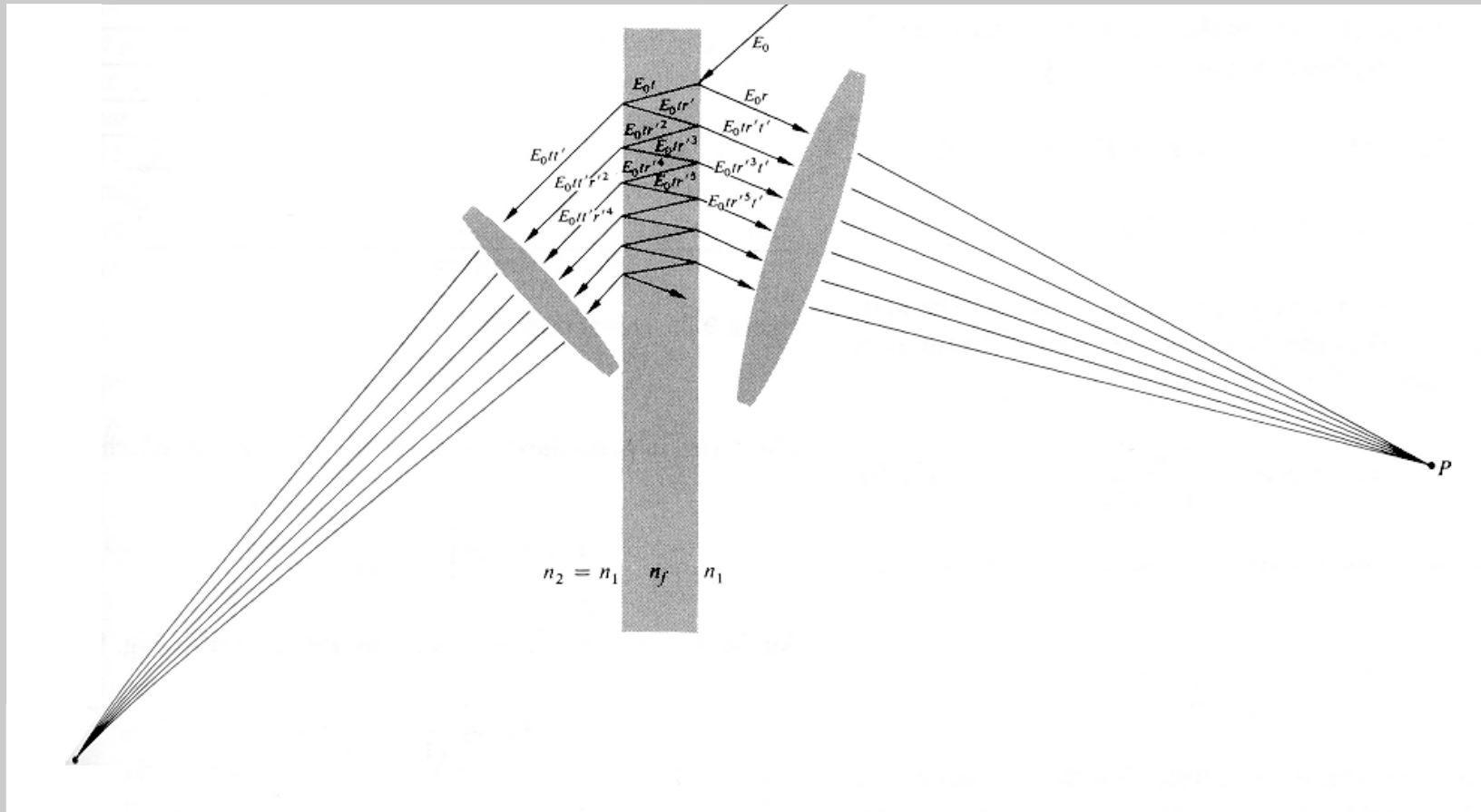
Multiple-beam interference: general formulation

Incident wave: E_0

Transmitted wave: E_{0t}

Reflected wave: E_{0r}

$n = 1$ n $n = 1$

$t\,t'\,e^{i\delta/2}\,E_0$

$t\,t'\,e^{i\delta/2}\left(r'\right)^2 e^{i\delta}\,E_0$

$t\,t'\,e^{i\delta/2}\left(\left(r'\right)^2 e^{i\delta}\right)^2 E_0$
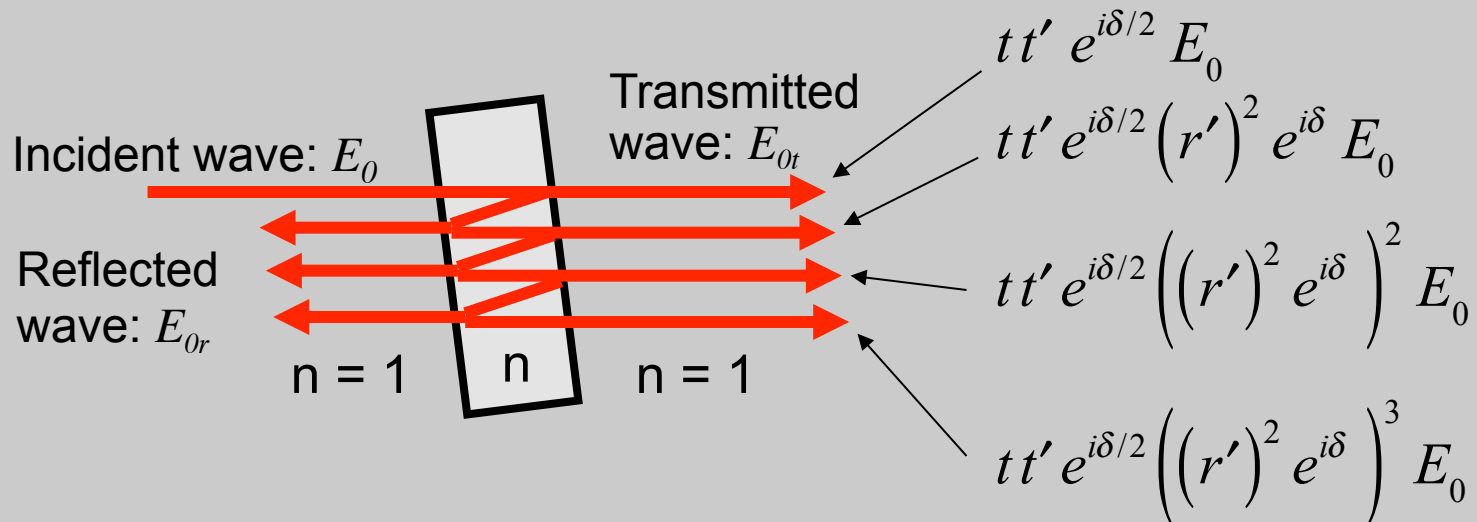
$t\,t'\,e^{i\delta/2}\left(\left(r'\right)^2 e^{i\delta}\right)^3 E_0$

r, t = reflection, transmission coefficients from air to glass

r', t' =   "   "   "   from glass to air

δ = round-trip phase delay inside medium = $k_0(2\,n\,L\,\cos\theta_t)$

Transmitted wave:

$$E_{0t} = tt'e^{-i\delta/2}E_0\left(1+\left(r'\right)^2 e^{i\delta}+\left(\left(r'\right)^2 e^{i\delta}\right)^2+\left(\left(r'\right)^2 e^{i\delta}\right)^3+\ldots\right)$$

Reflected wave:

$$E_{0r} = rE_0 + tt'r'e^{i\delta}E_0 + tt'r'\left(\left(r'\right)^2 e^{i\delta}\right)^2 E_0 + \ldots$$

Stokes Relations for reflection and transmission
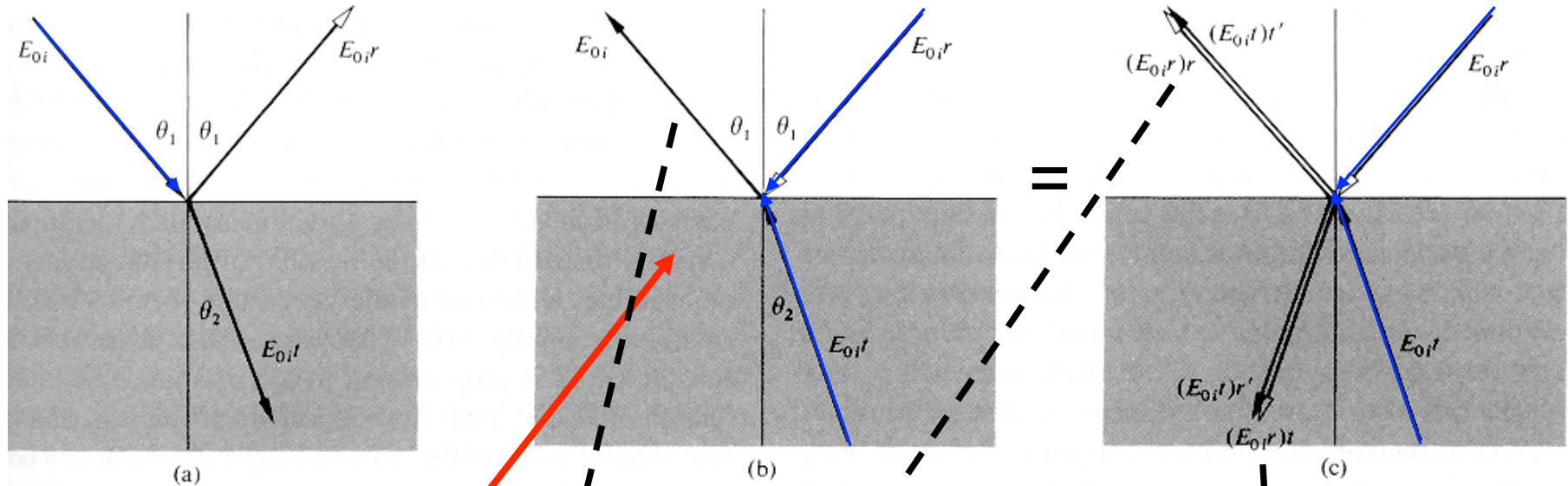
"Time reversal:"
Same amplitudes, reversed propagation direction

$$E_{oi} = (E_{oi}r)r + (E_{oi}t)t'$$

$$\therefore tt' = 1 - r^2$$

$$(E_{oi}t)r' + (E_{oi}r)t = 0$$

$$\therefore r' = -r$$

Notes:
• relations apply to angles connected by Snell's Law
• true for any polarization, but not TIR
• convention for which interface experiences a sign change can vary

Fabry-Perot transmission

$r' = -r$
$r'^2 = r^2$
$tt' = 1 - r^2$

The transmitted wave field is:

$$E_{0t} = tt' e^{i\delta/2} E_0 \left(1 + (r')^2 e^{i\delta} + \left((r')^2 e^{i\delta} \right)^2 + \left((r')^2 e^{i\delta} \right)^3 + \ldots \right)$$

$$= tt' e^{i\delta/2} E_0 \left(1 + r^2 e^{i\delta} + \left(r^2 e^{i\delta} \right)^2 + \left(r^2 e^{i\delta} \right)^3 + \ldots \right) = tt' e^{i\delta/2} E_0 \sum_{n=0}^{\infty} \left(r^2 e^{i\delta} \right)^n$$

Where:

$$\Rightarrow \qquad \boxed{E_{0t} = \frac{tt' e^{i\delta/2}}{1 - r^2 e^{-i\delta}} E_0}$$

$$(1 - x)^{-1} = 1 + x + x^2 + x^3 + \cdots$$

Power transmittance:

$$T \equiv \left| \frac{E_{0t}}{E_0} \right|^2 = \left| \frac{tt' e^{i\delta/2}}{1 - r^2 e^{i\delta}} \right|^2 = \frac{(tt')^2}{(1 - r^2 e^{+i\delta})(1 - r^2 e^{-i\delta})}$$

$$= \left[\frac{(tt')^2}{\{1 + r^4 - 2r^2 \cos\delta\}} \right] = \left[\frac{(1 - r^2)^2}{\{1 + r^4 - 2r^2[1 - 2\sin^2(\delta/2)]\}} \right] = \left[\frac{(1 - r^2)^2}{\{1 - 2r^2 + r^4 + 4r^2 \sin^2(\delta/2)]\}} \right]$$

Dividing numerator and denominator by $(1 - r^2)^2$

$$\boxed{T = \frac{1}{1 + F \sin^2(\delta/2)}} \qquad \text{where:} \qquad F = \left[\frac{2r}{1 - r^2} \right]^2$$

Multiple-beam interference: simple limits

Reflected waves

$$T = \frac{1}{1 + F \sin^2\left(\delta / 2\right)}$$

Full transmission: sin( ) = 0, d = 2 π m
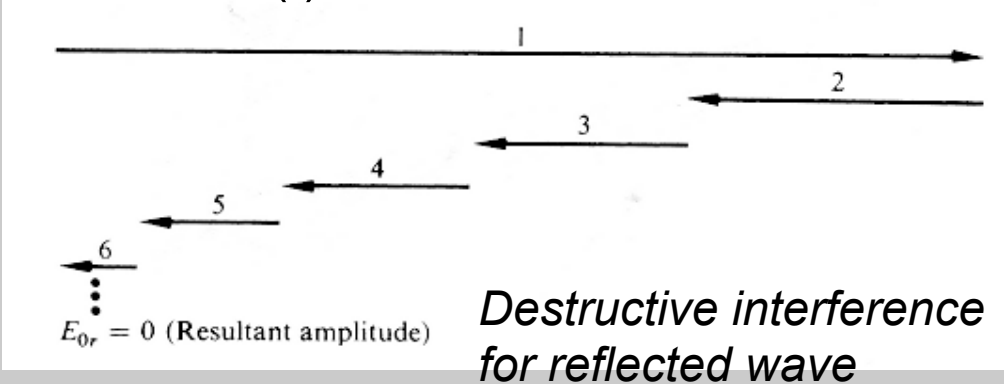
1st reflection

internal
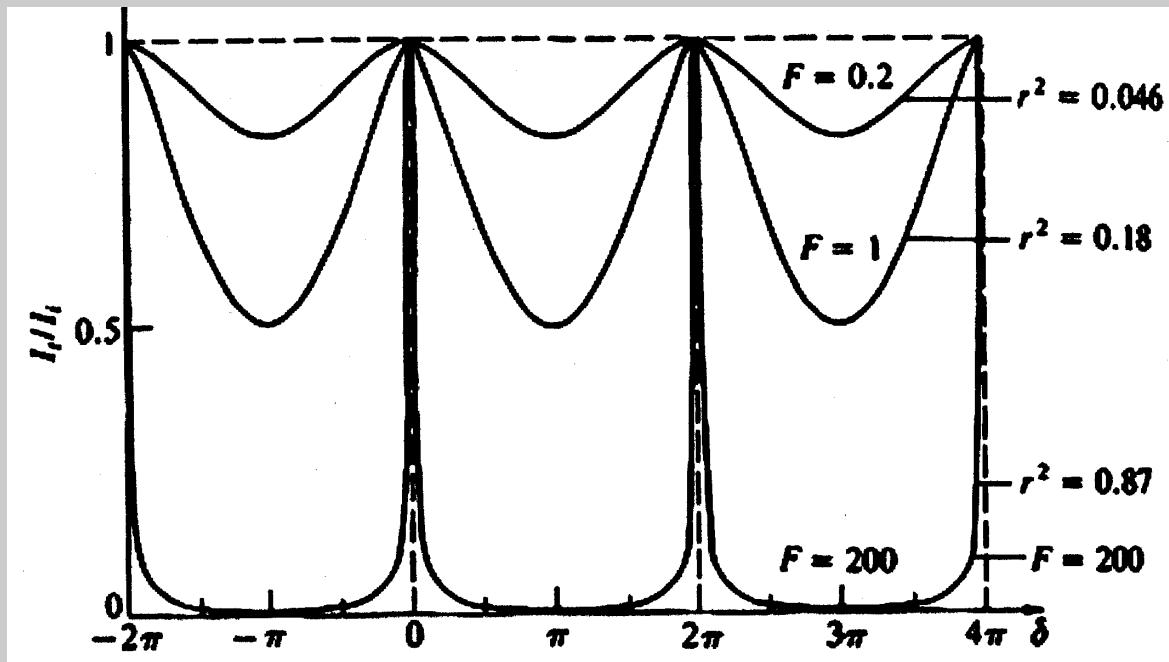reflections

*Destructive interference
for reflected wave*

Minimum transmission: sin( ) = 1, d = 2 π (m+1/2)
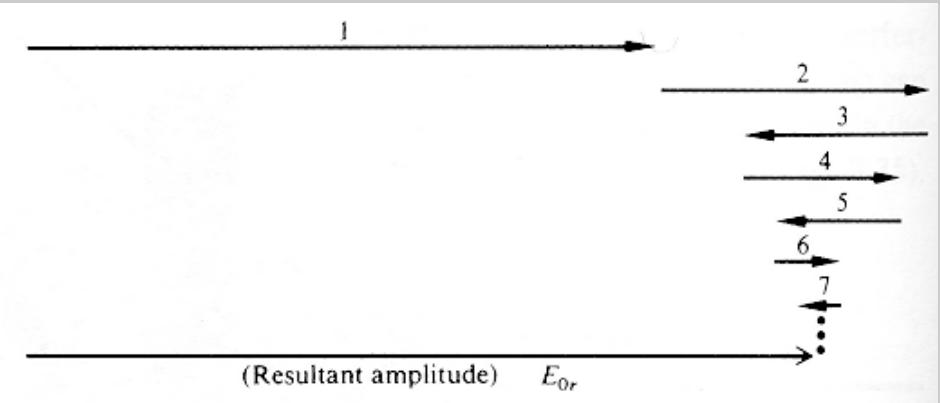
Constructive interference for reflected wave

Etalon transmittance vs. thickness, wavelength, or angle

$$T = \frac{1}{1 + F \sin^2 (\delta / 2)}$$

Transmission max:
sin( ) = 0, d = 2 π m

$$\delta = \frac{\omega}{c} 2 n L \cos [\theta_t]$$

$$= 2\pi m$$

At normal incidence:

$$\lambda_m = \frac{2 n L}{m} \quad \text{or} \quad n L = m \frac{\lambda_m}{2}$$

- The transmittance varies significantly with thickness or wavelength.
- We can also vary the incidence angle, which also affects δ.
- As the reflectance of each surface (R=r^2) approaches 1, the widths of the high-transmission regions become very narrow.

The Etalon Free Spectral Range

The Free Spectral Range is the wavelength range between transmission maxima.

λ_{FSR} =
Free Spectral Range
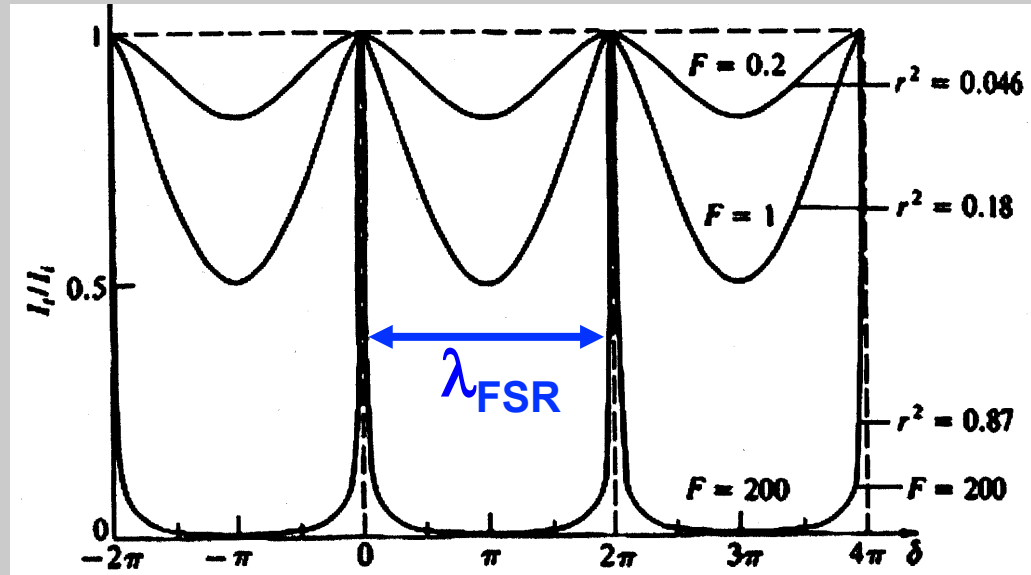
For neighboring orders:

$$\frac{4\pi nL}{\lambda_1} - \frac{4\pi nL}{\lambda_2} = 2\pi \Rightarrow \frac{1}{\lambda_1} - \frac{1}{\lambda_2} = \frac{1}{2nL} = \frac{\lambda_2 - \lambda_1}{\lambda_1 \lambda_2}$$

$$\lambda_2 - \lambda_1 = \lambda_{FSR}$$

$$\lambda_2 \lambda_1 \approx \lambda^2$$

$$\boxed{\lambda_{FSR} \approx \frac{\lambda^2}{2nL}}$$

$$\frac{\lambda_{FSR}}{\lambda} = \frac{\lambda}{2nL} = \frac{\nu_{FSR}}{\nu}$$

$$\boxed{\nu_{FSR} \approx \frac{c}{2nL}}$$

1/(round trip time)

Etalon Linewidth

The Linewidth δ_{LW} is a transmittance peak's full-width-half-max (FWHM).

$$T = \frac{1}{1 + F\sin^2\left(\delta/2\right)}$$

A maximum is where $\delta/2 \approx m\pi + \delta'/2$ and $\sin^2\left(\delta/2\right) \approx \delta'/2$

Under these conditions (near resonance),

$$T = \frac{1}{1 + F\delta'^2/4}$$

This is a Lorentzian profile, with FWHM at:

$$\frac{F}{4}\left(\frac{\delta_{LW}}{2}\right)^2 = 1 \implies \delta_{LW} \approx 4/\sqrt{F}$$

This transmission linewidth corresponds to the minimum resolvable wavelength.

Etalon Finesse ≈ resolution

The Finesse, $\mathfrak{I}$, is the ratio of the
Free Spectral Range and the Linewidth:

δ = 2π corresponds
to one FSR

$$\mathfrak{I} \equiv \frac{\delta_{FSR}}{\delta_{FW}} = \frac{2\pi}{4/\sqrt{F}} = \frac{\pi\sqrt{F}}{2}$$

Using: $\quad F = \left[\dfrac{2r}{1-r^2}\right]^2$

$$\boxed{\mathfrak{I} = \frac{\pi}{1-r^2}} \qquad \text{taking } r \approx 1$$

The Finesse is the number of wavelengths the interferometer can resolve.

Tools: scanning Fabry-Perot

Resonator with piezo control over mirror separation
http://www.thorlabs.us/newgrouppage9.cfm?objectgroup_id=859

- Wavelength range:
  535-820nm (ours)
- SA200 (ours)
  - FSR 1.5 GHz
  - Finesse > 200
  - Resolution 7.5MHz
- SA210
  - FSR 10 GHz
  - Finesse > 150
  - Resolution 67MHz
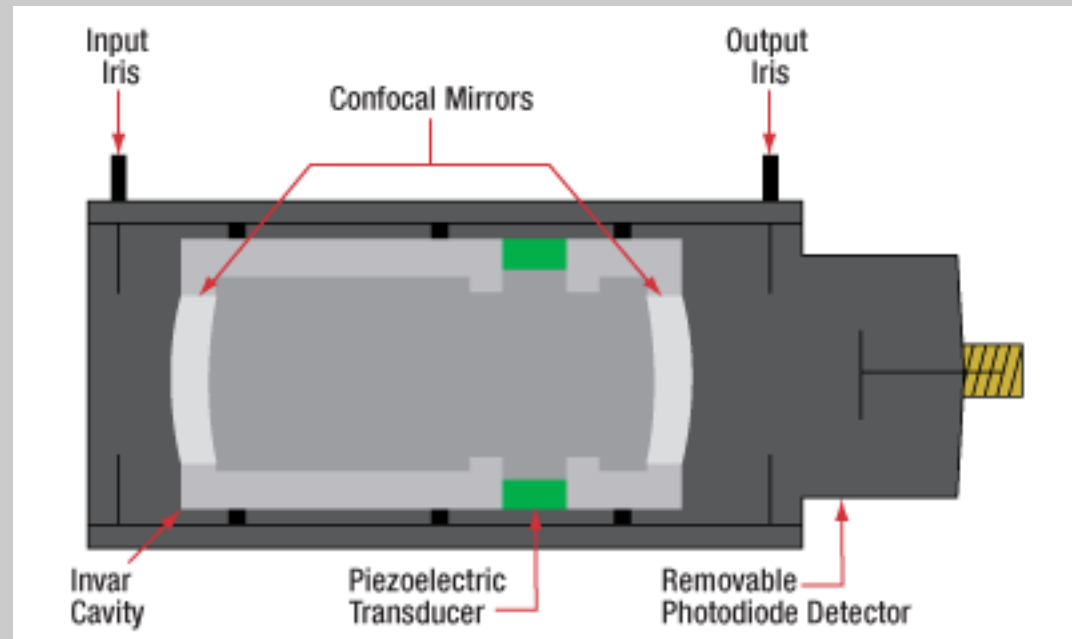
Tools: fixed plate Fabry-Perot
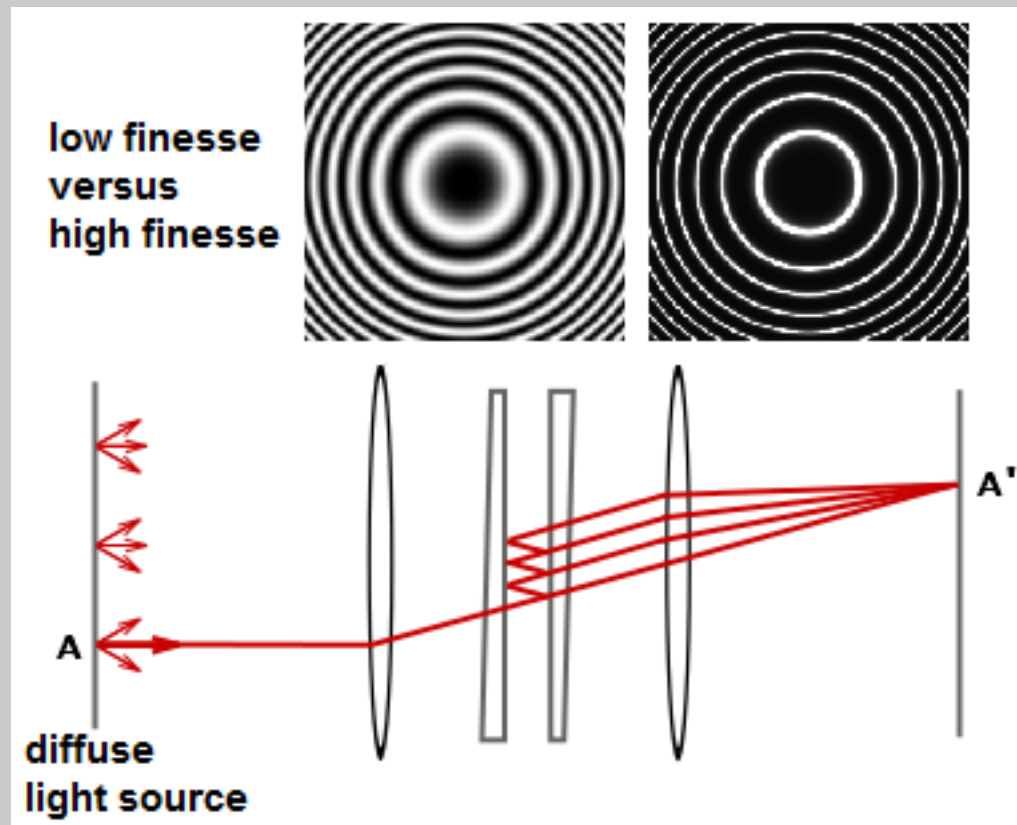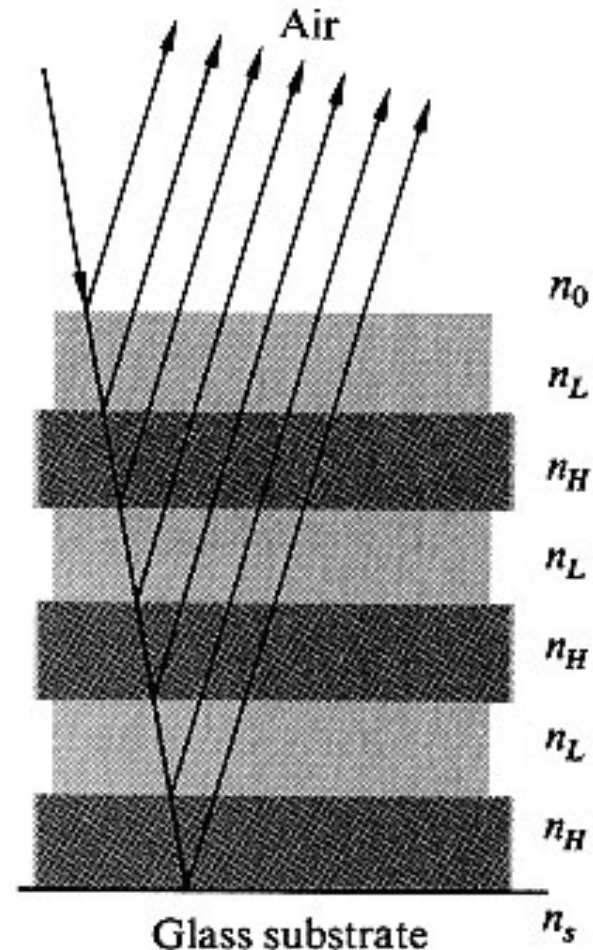
Multilayer coatings

Typical laser mirrors and camera lenses use many layers.
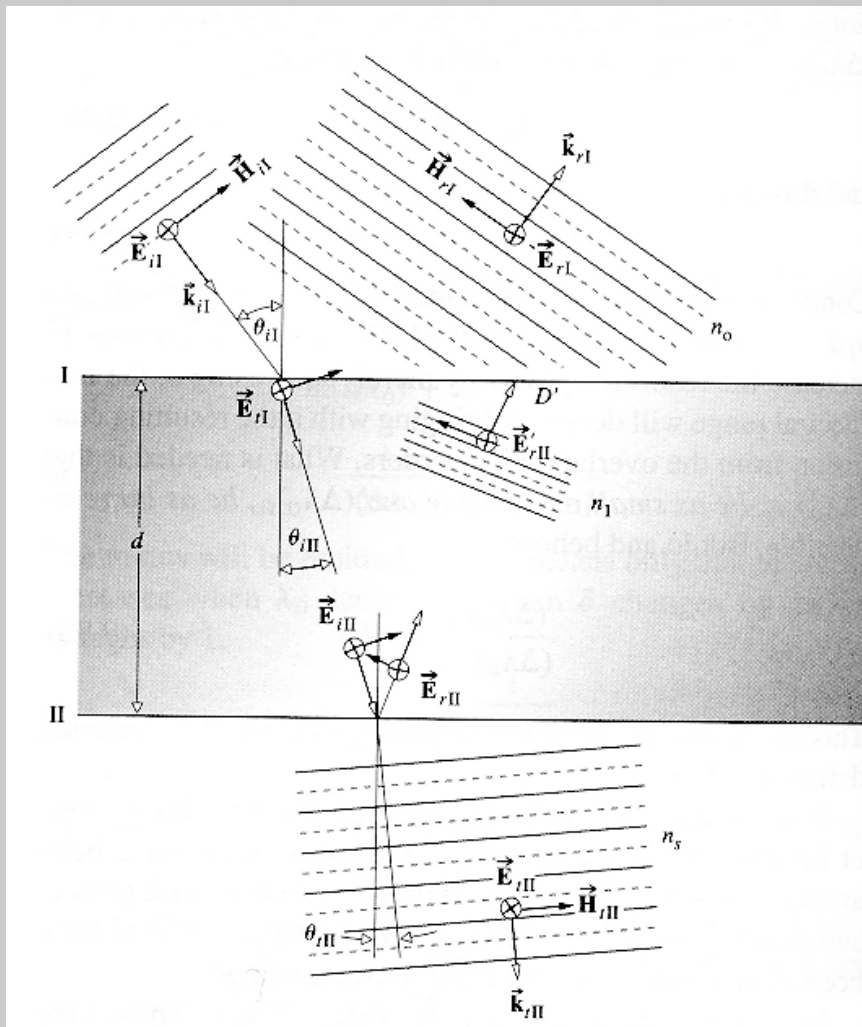
The reflectance and transmittance can be custom designed
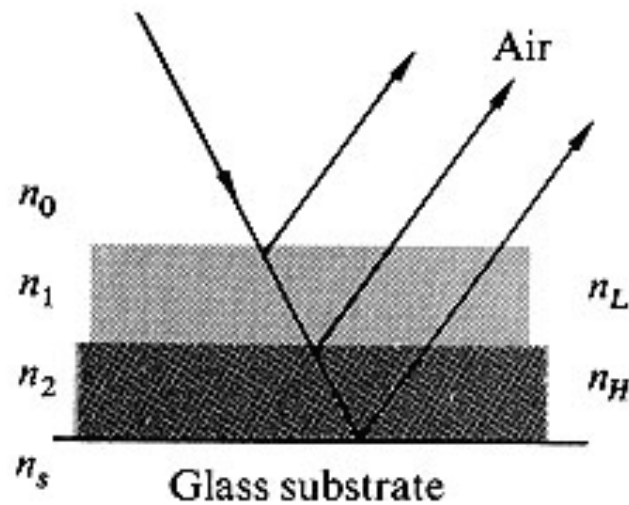
$g\,HL\,a$

Double-quarter

$g\,HL\,HL\,HL\,a$
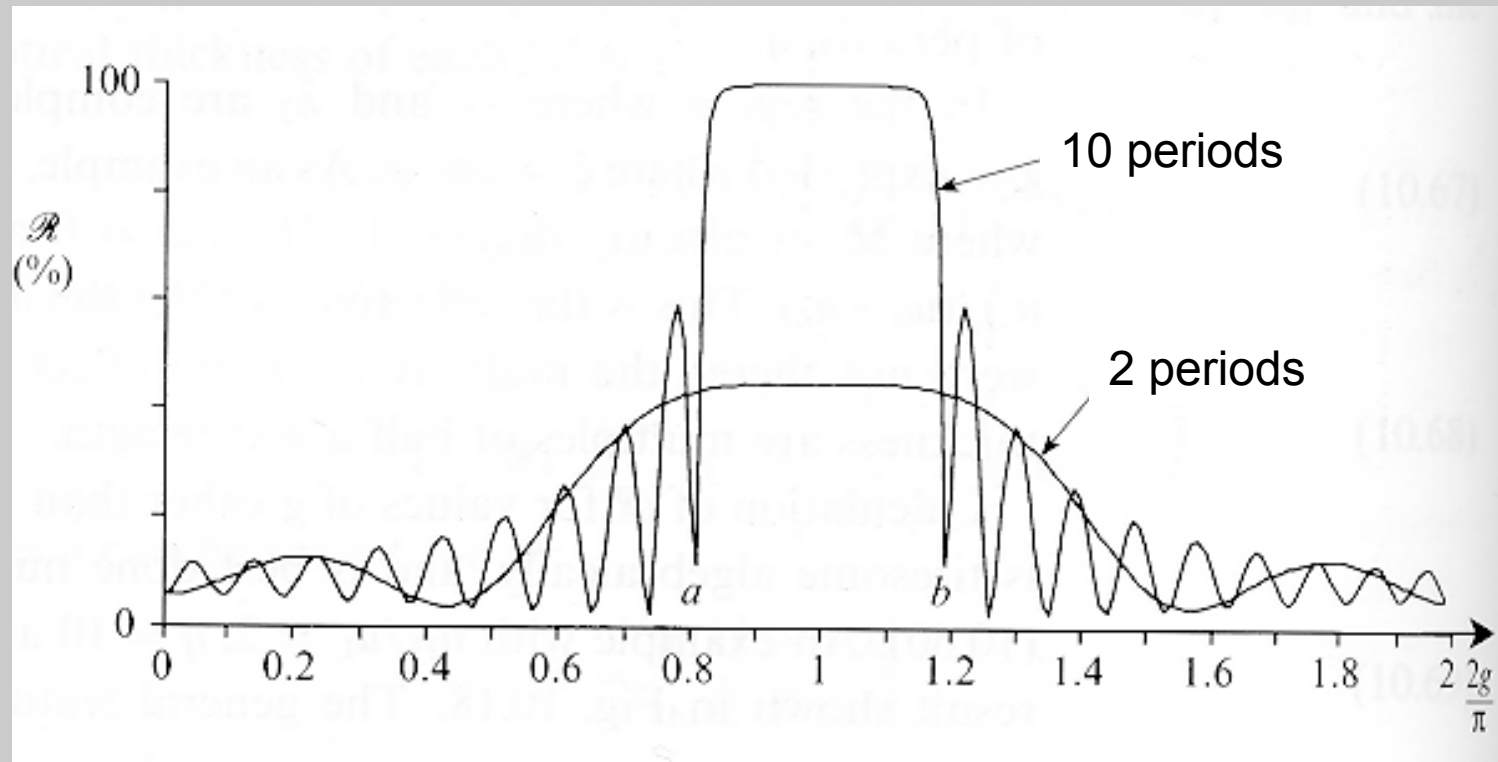$g(HL)^3a$

Quarter-wave stack

Multilayer thin-films: wave/matrix treatment

- Use boundary conditions to relate fields at the boundaries
- Phase shifts connect fields just after I to fields just before II
- Express this relation as a transfer matrix
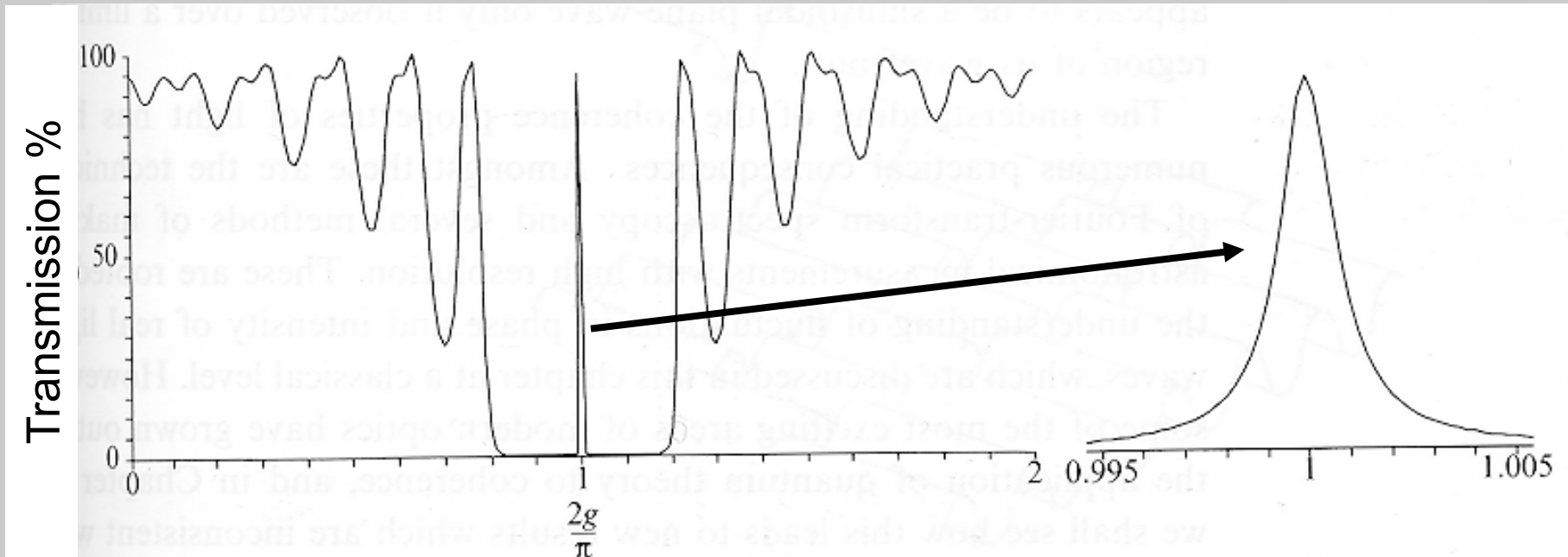- Multiply matrices for multiple layers

High-reflector design

Reflectivity can reach > 99.99% at a specific wavelength
> 99.5% for over 250nm
Bandwidth and reflectivity are better for "S" polarization.
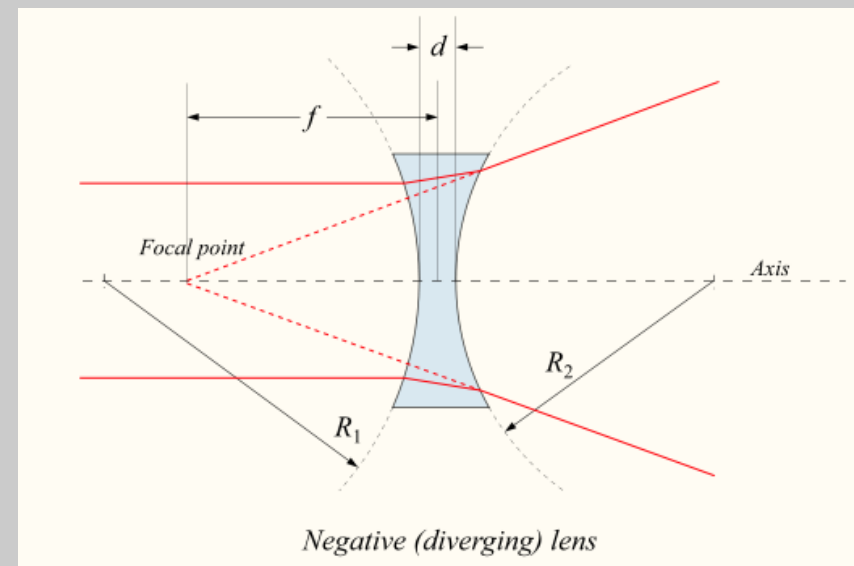
Interference filter design
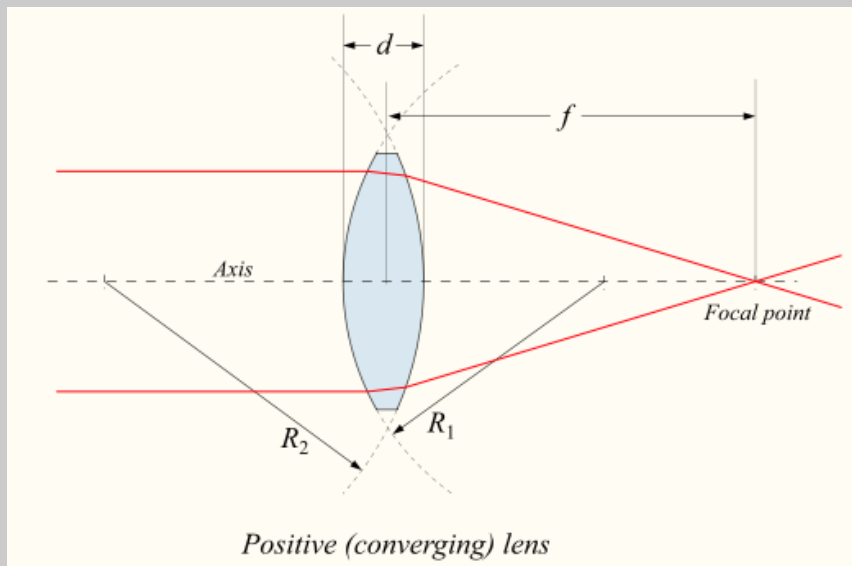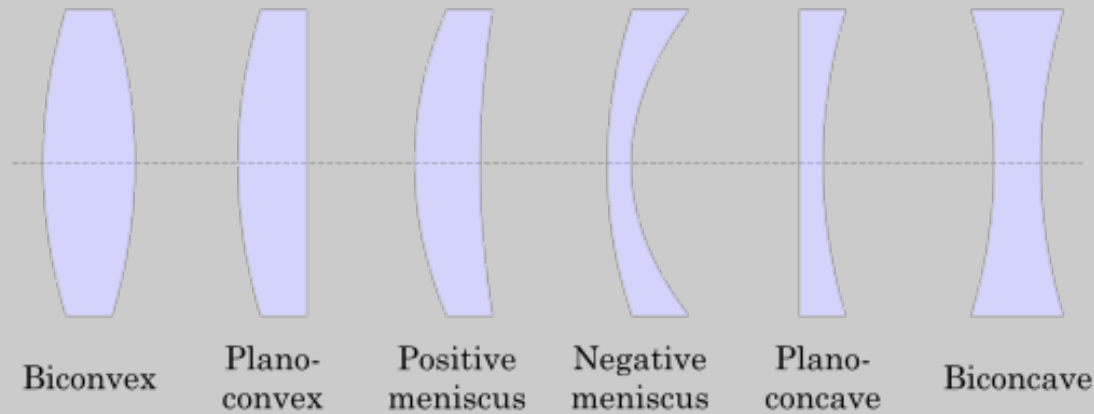

A thin layer is sandwiched between two high reflector coatings
-very large free spectral range, high finesse
- typically 5-10nm bandwidth, available throughout UV to IR
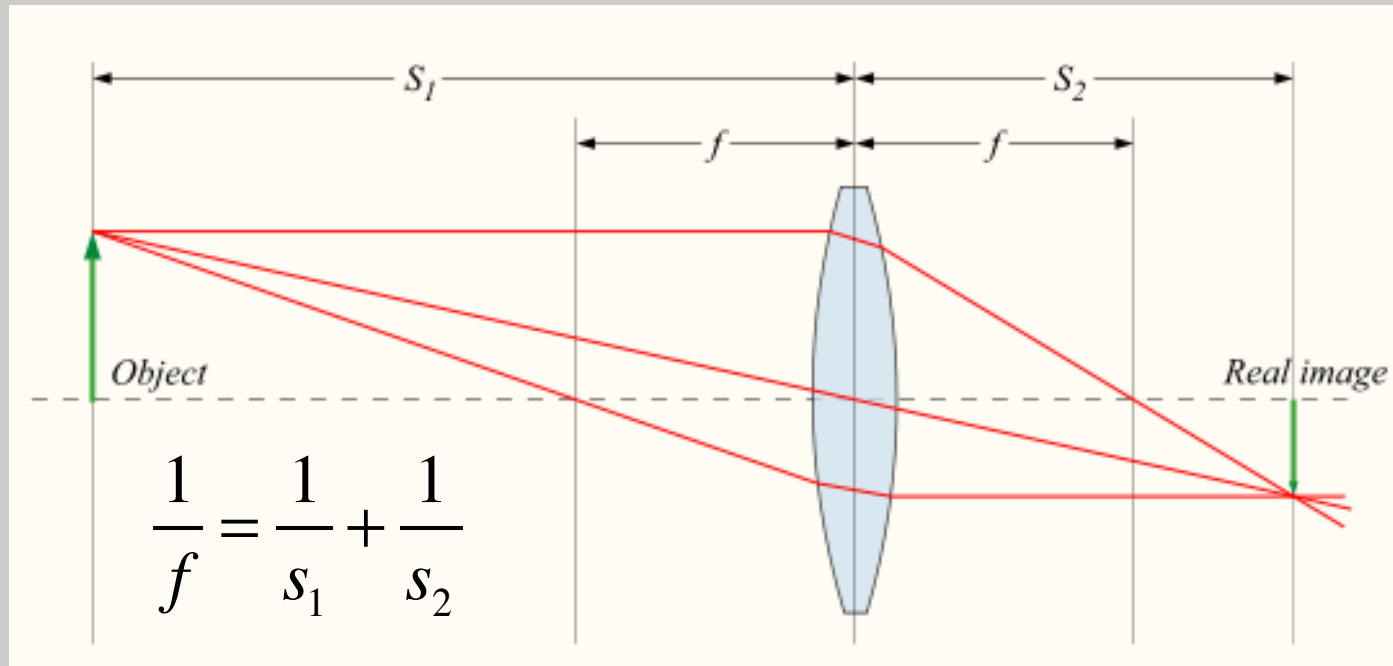
Geometric optics

- ## Approaches:
  - Paraxial calculations (assume small angle to optical axis)
    - Imaging equations: simple lens, Lensmakers' for thick lens
    - ABCD matrices for optical system
  - Non-paraxial, general case
    - Analytic calculation of aberrations: Zernike polynomials, Taylor-type expansions of wavefront error
    - Computer tracing (no approximations). Example: Zemax, Oslo

- ## Design procedure
  - Find existing design close to what could work for application
  - Paraxial trace with ray diagram
    - Calculate magnification, limiting apertures
  - Optimize with ABCD matrices or computer program
  - Analyze aberrations

Geometric optics: lenses

For a nice summary, see Lens (optics) Wikipedia page
https://en.wikipedia.org/wiki/Lens_(optics)

Biconvex Plano-convex Positive meniscus Negative meniscus Plano-concave Biconcave

Positive (converging) lens

Negative (diverging) lens

Geometric optics: imaging equation
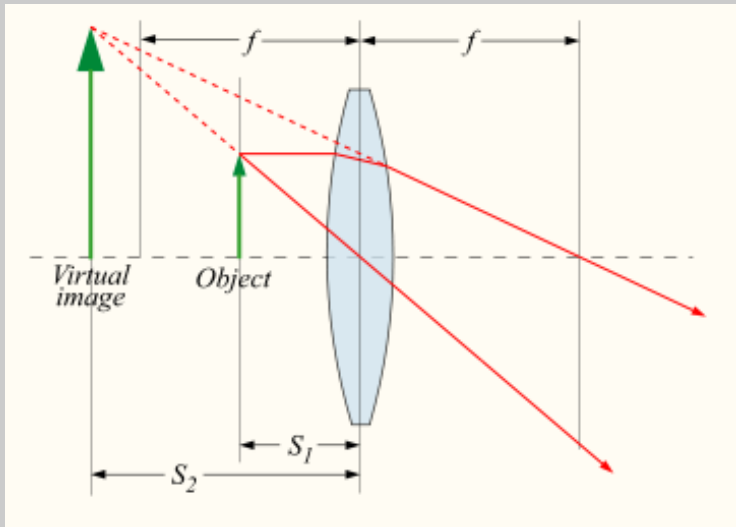

$$\frac{1}{f} = \frac{1}{s_1} + \frac{1}{s_2}$$

s_1, s_2 are positive as shown
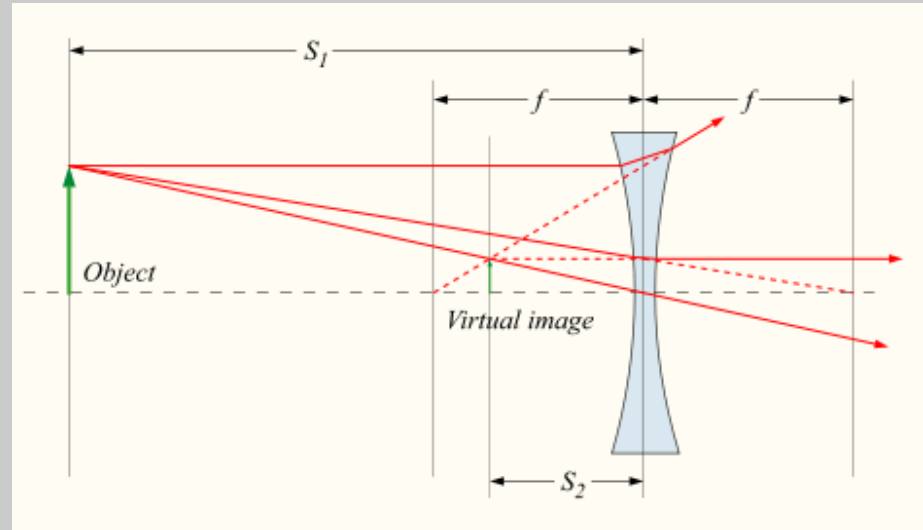s_1, s_2 = infinity: rays are collimated

$$M = -\frac{s_2}{s_1}$$

Magnification calculate with similar triangles
M < 0 inverted image
M = -1 when $s_1 = s_2 = 2\,f$, 1:1 imaging

Geometric optics: virtual images

s_1, $s_2 < 0$ if on the opposite side of lens

"virtual" image: position where image seems to come from

Concave lens: f < 0
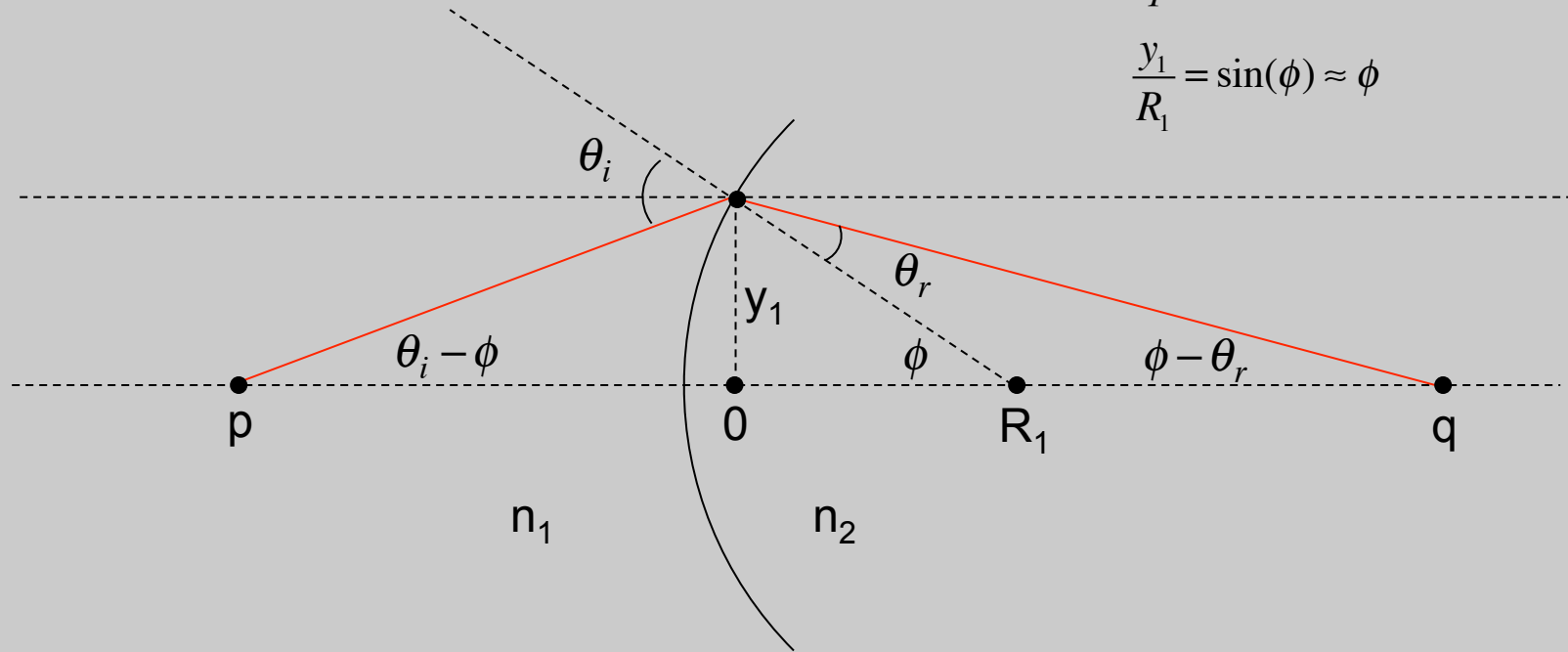
Same convention for s_1, s_2

Raytracing: single curved interface

Snell: $n_1 \sin(\theta_i) = n_2 \sin(\theta_r)$

$$\frac{y_1}{p} = \tan(\theta_i - \phi) \approx \theta_i - \phi$$

$$\frac{y_1}{q} = \tan(\phi - \theta_r) \approx \phi - \theta_r$$

$$\frac{y_1}{R_1} = \sin(\phi) \approx \phi$$

θ_i

θ_r

y_1

$\theta_i - \phi$

ϕ

$\phi - \theta_r$

p

0

R_1

q

n_1

n_2

$$\frac{n_2}{n_1} = \frac{\sin(\theta_i)}{\sin(\theta_r)} \approx \frac{\theta_i}{\theta_r} = \frac{\phi + \frac{y_1}{p}}{\phi - \frac{y_1}{q}} = \frac{\frac{y_1}{R_1} + \frac{y_1}{p}}{\frac{y_1}{R_1} - \frac{y_1}{q}} = \frac{\frac{1}{R_1} + \frac{1}{p}}{\frac{1}{R_1} - \frac{1}{q}}$$

In paraxial appx, y's cancel

$$n_2\left(\tfrac{1}{R_1} - \tfrac{1}{q}\right) = n_1\left(\tfrac{1}{R_1} + \tfrac{1}{p}\right) \rightarrow \tfrac{1}{R_1}(n_2 - n_1) = \tfrac{n_2}{q} + \tfrac{n_1}{p}$$

Raytracing: two curved interfaces

- add second interface: R > 0 if center is to right
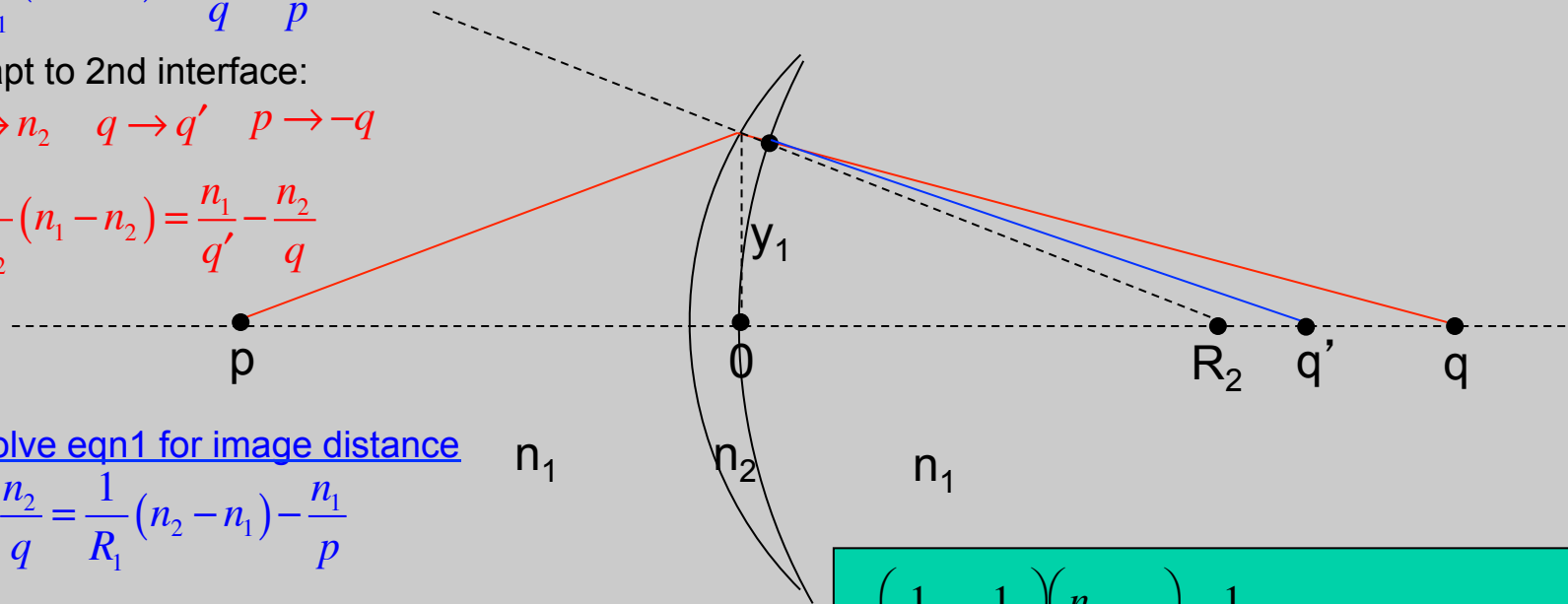- assume $y_2 = y_1$

Eqn from 1st:

$$\frac{1}{R_1}(n_2 - n_1) = \frac{n_2}{q} + \frac{n_1}{p}$$

Adapt to 2nd interface:

$$n_1 \leftrightarrow n_2 \quad q \rightarrow q' \quad p \rightarrow -q$$

$$\rightarrow \frac{1}{R_2}(n_1 - n_2) = \frac{n_1}{q'} - \frac{n_2}{q}$$

n_1 n_2 n_1

y_1

p   0 R_2 q' q

Solve eqn1 for image distance

$$\rightarrow \frac{n_2}{q} = \frac{1}{R_1}(n_2 - n_1) - \frac{n_1}{p}$$

$$\frac{1}{R_2}(n_1 - n_2) = \frac{n_1}{q'} - \frac{1}{R_1}(n_2 - n_1) + \frac{n_1}{p}$$

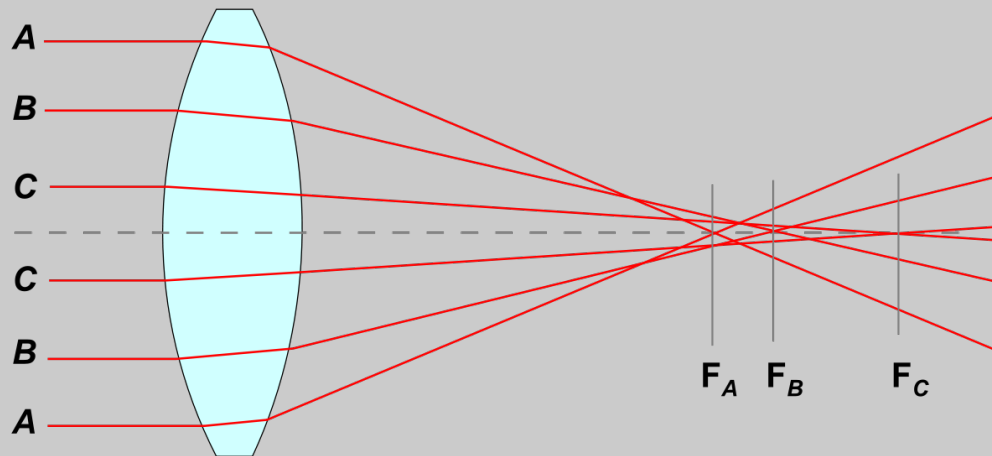$$\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\left(\frac{n_2}{n_1} - 1\right) = \frac{1}{q'} + \frac{1}{p}$$

$$\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\left(\frac{n_2}{n_1} - 1\right) = \frac{1}{f} \quad \text{Focal length (lensmaker's eqn)}$$

$$\frac{1}{f} = \frac{1}{s_o} + \frac{1}{s_i} \quad \text{Imaging equation}$$
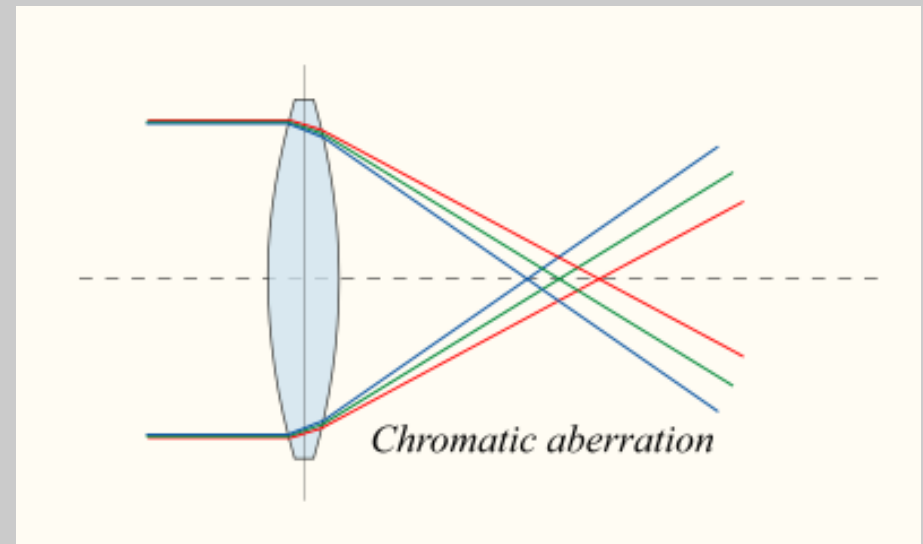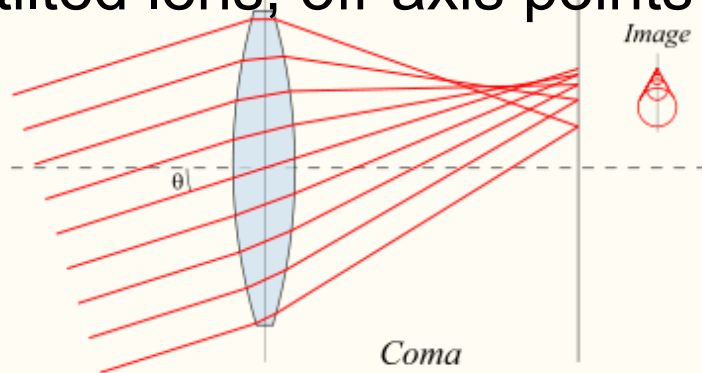
Aberrations

- For non-paraxial rays errors in how rays focus

Spherical aberration

Coma:
tilted lens, off-axis points

Coma

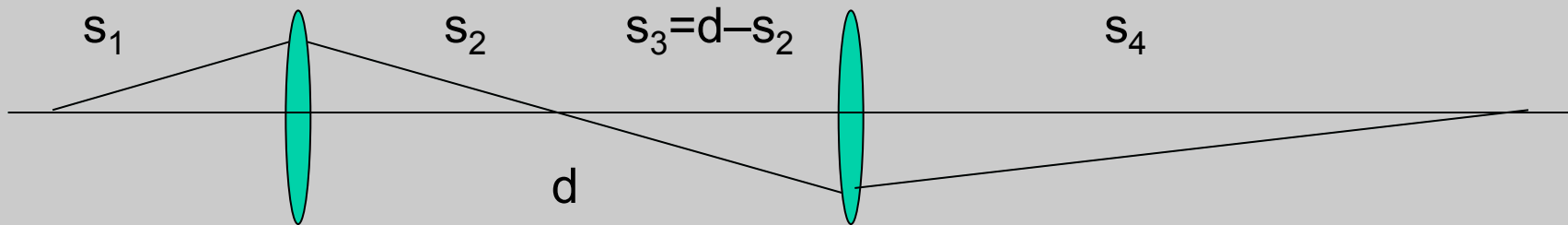Chromatic aberration

Lens systems

- It is possible to cascade the lens imaging equation for multiple lenses:
    - Image from lens 1 is object for lens 2

s_1 s_2 $s_3 = d - s_2$ s_4

d

- For more complicated systems, use a matrix method: ABCD matrices
    - Also good for resonators
    - Does not account for aberrations

ABCD ray matrices

- Formalism to propagate rays through optical systems
  - Keep track of ray height r and ray angle $\theta = dr/dz = r'$
  - Treat this pair as a vector: $\begin{pmatrix} r \\ r' \end{pmatrix}$

  - Optical system will modify both the ray height and angle, e.g.

  $$\begin{pmatrix} r_2 \\ r_2' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} r_1 \\ r_1' \end{pmatrix}$$

  - Successive ABCD matrices multiply from the left

- Translation

  $$r_2 = r_1 + Lr_1'$$
  $$r_2' = r_1'$$
  $$\rightarrow \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$
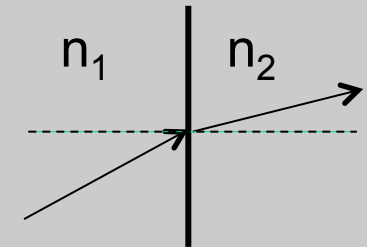
  $r_1 \qquad\qquad\qquad r_2$
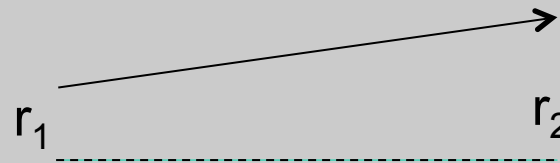
Refraction in ABCD

- Translation: $\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$

- Flat interface

$$r_2 = r_1 \qquad n_1 \sin\theta_1 = n_2 \sin\theta_2 \quad \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & n_1/n_2 \end{pmatrix}$$

$$n_1\theta_1 \approx n_2\theta_2$$

$$r_2' = \frac{n_1}{n_2} r_1'$$

Special case:
$n_1 = 1,\ n_2 = n$ $\rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1/n \end{pmatrix}$

n_1 | n_2

- Window: calculate matrix
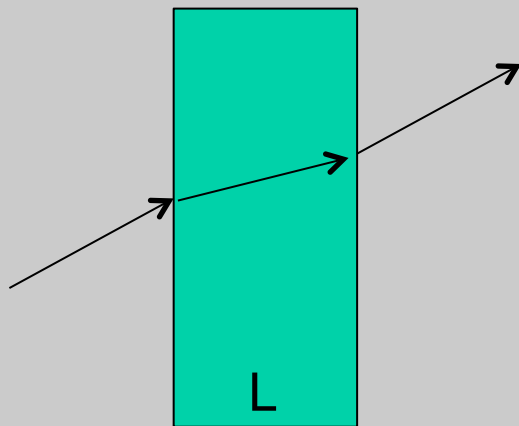
$$\rightarrow \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1/n \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} \begin{pmatrix} 1 & L/n \\ 0 & 1/n \end{pmatrix} = \begin{pmatrix} 1 & L/n \\ 0 & 1 \end{pmatrix}$$

L

Effective thickness reduced by n

Curved surfaces in ABCD

- Thin lens: matrix computes transition from one side of lens to other

$$r_2 = r_1$$

$$r_1' = r_1 / p$$

$$r_2' = -r_1 / q$$

$$\frac{1}{q} = \frac{1}{f} - \frac{1}{p} \qquad \rightarrow r_2' = -r_1\left(\frac{1}{f} - \frac{1}{p}\right) = -\frac{r_1}{f} + r_1' \qquad \text{lens}\left(\begin{array}{cc} A & B \\ C & D \end{array}\right) \rightarrow \left(\begin{array}{cc} 1 & 0 \\ -1/f & 1 \end{array}\right)$$

- Spherical interface: radius *R*

$$\rightarrow \left(\begin{array}{cc} 1 & 0 \\ \dfrac{n_1 - n_2}{n_2}\dfrac{1}{R} & \dfrac{n_1}{n_2} \end{array}\right)$$

3D wave propagation

$$\nabla^2 \mathbf{E} - \frac{n_j^2}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E} = \frac{\partial^2}{\partial z^2}\mathbf{E} + \nabla_\perp^2 \mathbf{E} - \frac{n(\mathbf{r})^2}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E} = 0$$

$$\nabla_\perp^2 = \partial_x^2 + \partial_y^2 \qquad \boxed{\nabla_\perp^2 = \frac{1}{r}\partial_r\left(r\,\partial_r\right) + \frac{1}{r^2}\partial_\phi^2}$$

- Note:
  - All linear propagation effects are included in LHS: diffraction, interference, focusing…
  - Previously, we assumed plane waves where transverse derivatives are zero.

- More general examples:
  - Gaussian beams (including high-order)
  - Waveguides
  - Arbitrary propagation
  - Can determine discrete solutions to linear equation (e.g. Gaussian modes, waveguide modes), then express fields in terms of those solutions.

Paraxial, slowly-varying approximations

- Assume
  - waves are forward-propagating:

    $$\mathbf{E}(\mathbf{r},t) = \mathbf{A}(\mathbf{r})e^{i(kz-\omega_0 t)} + \text{c.c.}$$

  - Refractive index is isotropic

    $$\frac{\partial^2}{\partial z^2}\mathbf{A} + 2ik\frac{\partial}{\partial z}\mathbf{A} - k^2\mathbf{A} + \nabla_\perp^2\mathbf{A} + \frac{n^2\omega_0^2}{c^2}\mathbf{A} = 0$$

  - Fast oscillating carrier terms cancel (blue)
- Slowly-varying envelope: compare red terms
  - Drop 2nd order deriv if $\quad \dfrac{2\pi}{\lambda}\dfrac{1}{L}A \gg \dfrac{1}{L^2}A$

  - This ignores:
    - Changes in z as fast as the wavlength
    - Counterpropagating waves

Gaussian beam solutions to wave equation

- Without any source term, paraxial equation is

$$2ik\frac{\partial}{\partial z}\mathbf{A} + \nabla_\perp^2 \mathbf{A} = 0$$

- Gaussian beam solutions can be written as:

$$A(r,z) = A_0 \frac{1}{1+i\xi} e^{-\frac{r^2}{w_0^2(1+i\xi)}} \qquad \xi = \frac{z}{z_R} \qquad z_R = n\frac{\pi w_0^2}{\lambda} = \frac{k w_0^2}{2}$$

Rayleigh range

$$E(r,z,t) = A_0 \frac{w_0}{w(z)} e^{i(kz-\omega t)} e^{-\frac{r^2}{w^2(z)}} e^{i\frac{kr^2}{2R(z)}} e^{-i\eta(z)}$$
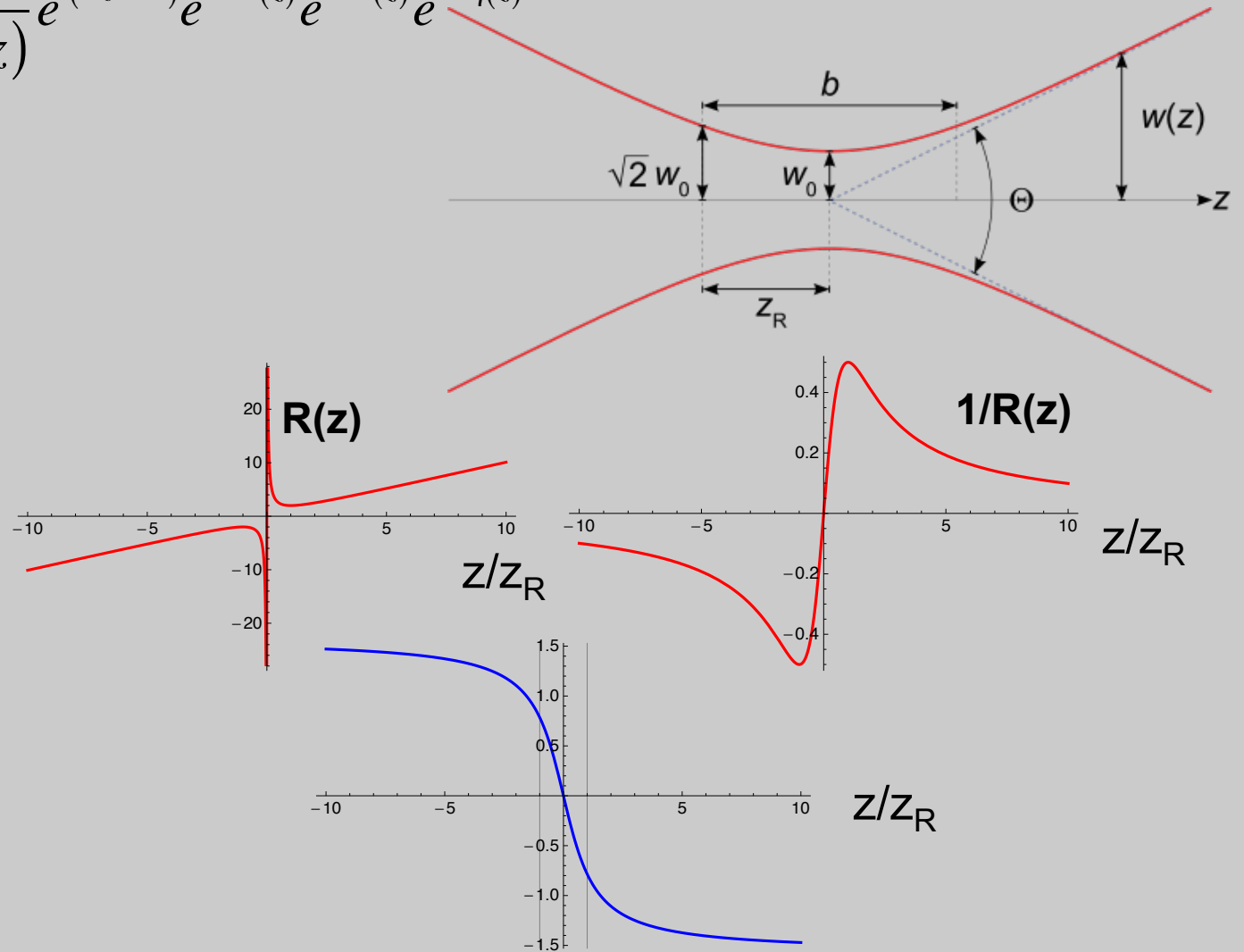
Standard form of Gaussian beam equations

$$E\left(r,z,t\right) = A_0 \, \frac{w_0}{w(z)} \, e^{i(kz-\omega t)} \, e^{-\frac{r^2}{w^2(z)}} \, e^{i\frac{kr^2}{2R(z)}} \, e^{-i\eta(z)}$$

$$w(z) = w_0 \sqrt{1 + \frac{z^2}{z_R^2}}$$

$$R(z) = z\left(1 + \frac{z_R^2}{z^2}\right)$$

Gouy phase

$$\eta(z) = \arctan\left(\frac{z}{z_R}\right)$$

Complex q form for Gaussian beam

- This combines beam size and radius of curvature into one complex parameter
  - This form is used for for ABCD calculations

$$A(r,z) = A_0 \frac{1}{1+i\xi} e^{-\frac{r^2}{w_0^2(1+i\xi)}} \qquad \rightarrow A(r,z) = \frac{1}{q(z)} e^{-ik\frac{r^2}{2q(z)}}$$

$$\frac{1}{q(z)} = \frac{1}{R(z)} - i\frac{\lambda}{\pi w^2(z)}$$

$$\frac{1}{q(z)} = \frac{1}{z+iz_R} = \frac{z}{z^2+z_R^2} - i\frac{z_R}{z^2+z_R^2}$$

$$\frac{1}{R(z)} = \frac{1}{z\left(1+\frac{z_R^2}{z^2}\right)} = \frac{z}{z^2+z_R^2}$$

$$= \frac{1}{R(z)} - i\frac{w_0^2}{z_R w^2(z)}$$

$$= \frac{1}{R(z)} - i\frac{\lambda}{\pi w^2(z)} = \frac{1}{R(z)} - i\frac{1}{Z(z)}$$

$$\frac{1}{w^2(z)} = \frac{1}{w_0^2\left(1+\frac{z^2}{z_R^2}\right)} = \frac{z_R^2}{w_0^2(z^2+z_R^2)}$$

Complex q vs standard form

$$u(r,z) = \frac{1}{q(z)} e^{-ik\frac{r^2}{2q(z)}} \quad \text{with} \quad \frac{1}{q(z)} = \frac{1}{R(z)} - i\frac{\lambda}{\pi w^2(z)}$$

Expand exponential:

$$\exp\left[-ik\frac{r^2}{2q(z)}\right] = \exp\left[-ik\frac{r^2}{2}\left(\frac{1}{R(z)} - i\frac{\lambda}{\pi w^2(z)}\right)\right]$$

$$= \exp\left[-ik\frac{r^2}{2}\frac{1}{R(z)} - i\frac{2\pi}{\lambda}\frac{r^2}{2}\left(-i\frac{\lambda}{\pi w^2(z)}\right)\right] = e^{-ik\frac{r^2}{2R(z)}} e^{-\frac{r^2}{w^2(z)}}$$

$$a + ib = \sqrt{a^2 + b^2}\, e^{i\arctan(b/a)}$$

Expand leading inverse q:

$$\frac{1}{q(z)} = -i\left(\frac{iz}{z^2 + z_R^2} + \frac{z_R}{z^2 + z_R^2}\right) = -i\left(\frac{\sqrt{z^2 + z_R^2}}{z^2 + z_R^2}\right) e^{i\arctan(z/z_R)}$$

$$= -i\left(\frac{1}{z_R\sqrt{1 + z^2/z_R^2}}\right) e^{i\arctan(z/z_R)} = \frac{w_0}{i z_R w(z)} e^{i\eta(z)}$$

Difference between Siegman's complex q and standard form

$$u(r,z) = \frac{1}{q(z)} e^{-ik\frac{r^2}{2q(z)}} = \frac{1}{iz_R}\frac{w_0}{w(z)} e^{i\eta(z)} e^{-ik\frac{r^2}{2R(z)}} e^{-\frac{r^2}{w^2(z)}}$$

$$E(r,z,t) = A_0 \frac{w_0}{w(z)} e^{i(kz-\omega t)} e^{-\frac{r^2}{w^2(z)}} e^{i\frac{kr^2}{2R(z)}} e^{-i\eta(z)}$$

- Siegman's form for the complex q is used almost everywhere for the ABCD calculations.

- He uses the exp[+ I w t] convention, which accounts for the sign difference in the complex exponentials.

- With exp[-I w t] convention, define q as:

$$\frac{1}{q(z)} = \frac{1}{R(z)} + i\frac{\lambda}{\pi w^2(z)} = \frac{1}{z - iz_R}$$

Compare Boyd's form to standard:

- Boyd's complex form is consistent with standard Gaussian beam form

$$A(r,z) = A_0 \frac{1}{1+i\xi} e^{-\frac{r^2}{w_0^2(1+i\xi)}} = A_0 \frac{1}{1+iz/z_R} e^{-\frac{r^2}{w_0^2(1+iz/z_R)}}$$

$$\frac{1}{1+i\xi} = \frac{1}{1+iz/z_R} = \frac{z_R}{z_R+iz} = \frac{-iz_R}{z-iz_R} = \frac{-iz_R}{q(z)}$$

$$A(r,z) = A_0 \left(-iz_R\right) \frac{1}{q(z)} e^{+\frac{iz_R r^2}{w_0^2 q(z)}} = -iz_R A_0 \frac{1}{q(z)} e^{+\frac{ikr^2}{2q(z)}}$$