Test 1 Solutions

Note Title 2/18/2008

1) a) A is Hermitian iff $A^+ = A$
where $A^+ = (A^T)^*$
or
$(A\varphi, \psi) = (\varphi, A\psi)$ for all ψ, φ

b) i) eigenvalues are real
ii) eigenvectors associated with distinct eigenvalues are orthogonal
iii) eigenvectors form a complete set

2 Since the H is so light we can assume the CL is effectively at rest

$$w^2 = \frac{k}{m} \approx \frac{500 \text{ N/m}}{2 \times 10^{-27} \text{ kg}} = 250 \times 10^{28} \left(\frac{\text{radians}}{\text{sec}}\right)^2$$

$$w \approx 15 \times 10^{14}$$

$$\Rightarrow f \approx 2.5 \times 10^{14} \text{ Hz}$$

$$\hbar w = (1 \times 10^{-34} \text{ Js})(15 \times 10^{14} \text{ 1/s})$$

$$= 15 \times 10^{-20} \text{ J} \cdot 6 \times 10^{18} \frac{\text{J}}{\text{eV}}$$

$$\cong .9 \text{ eV}$$

4 $\quad \psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar} x^2}$

Let $\quad \frac{m\omega}{\hbar} = \alpha$

$$\psi_0 = \pi^{-1/4} \alpha^{1/4} e^{-\alpha/2 x^2}$$

$$a = \frac{1}{\sqrt{2}} \sqrt{\alpha} \left(x + \frac{i}{m\omega} p\right)$$

$$a\psi_0 = \pi^{-1/4} \frac{1}{\sqrt{2}} \alpha^{3/4} \underbrace{\left\{\left(x + \frac{i}{m\omega}\left(-i\hbar \frac{d}{dx}\right)\right)\right\} e^{-\alpha/2 x^2}}$$

$$x e^{-\alpha/2 x^2} + \frac{1}{\alpha} \cdot (-\alpha) x e^{-\alpha/2 x^2} \equiv 0$$

---

5) $\quad \psi(x,t) = A e^{-a\left[m\frac{x^2}{\hbar} + it\right]}$

$$|\psi(x,t)|^2 = |A|^2 e^{-2am/\hbar x^2}$$

$$1 = A^2 \underbrace{\int_{-\infty}^{\infty} e^{-\alpha x^2} dx}_{= \sqrt{\pi/\alpha}} \qquad \alpha = 2am/\hbar$$

$$\Rightarrow A^2 \sqrt{\pi/\alpha} = 1$$

$$\Rightarrow A = \left(\frac{\alpha}{\pi}\right)^{1/4} = \left(\frac{2am}{\pi\hbar}\right)^{1/4}$$

$$\psi(x,t) = \left(\frac{2am}{\pi\hbar}\right)^{1/4} e^{-\frac{am}{\hbar}x^2} e^{-iat}$$

$\langle x \rangle = 0$

$\langle x^2 \rangle = \dfrac{\hbar}{4am} = \sigma_x^2$

$\langle p \rangle = 0$

$\langle p^2 \rangle = a\hbar m = \sigma_p^2$

$$\boxed{\sigma_x \sigma_p = \frac{\hbar}{2}}$$

6) $E_n = \dfrac{n^2 \pi^2 \hbar^2}{2ma^2} = \dfrac{n^2 \pi^2 \hbar^2}{2}$

$E_1 = \text{ground state} = \dfrac{\pi^2 \hbar^2}{2}$

so $\psi(x,0) = \sqrt{.9}\,\psi_1(x) + \sqrt{.1}\,\psi_2(x)$

$= \sqrt{.9}\,\sqrt{2}\sin(\pi x) + \sqrt{.1}\,\sqrt{2}\sin(2\pi x)$

$\psi(x,t) = \sqrt{2}\left\{\sqrt{.9}\sin(\pi x)e^{-i\frac{\pi^2\hbar}{2}t} + \sqrt{.1}\sin(2\pi x)\,e^{-i2\pi^2\hbar t}\right\}$

we know that

7) $H\psi_n = (n+1/2)\hbar\omega\,\psi_n$

we showed that $H = \hbar\omega(a^+a + \frac{1}{2})$

$\Rightarrow \dfrac{1}{\hbar\omega}H - \dfrac{1}{2} = a^+a$

So $a^+ a \psi_N = \frac{1}{\hbar\omega} H\psi_N - \frac{1}{2}\psi_N$

$$= \frac{1}{\hbar\omega}\left(N+\tfrac{1}{2}\right)\hbar\omega\, \psi_N - \tfrac{1}{2}\psi_N$$

$$= \left(N+\tfrac{1}{2}-\tfrac{1}{2}\right)\psi_N$$

$$= N\psi_N$$

This is why a^+a is called the number operator.

$[H,x] = Hx - xH$

apply to test fcn

$Hx\psi - xH\psi$

or you could use a specific potential

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V\right)x\psi - x\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V\right)\psi$$

$$= -\frac{\hbar^2}{2m}\left\{\frac{d}{dx}\left[\psi + x\psi'\right]\right\} + \frac{\hbar^2}{2m}x\psi''$$

$$= -\frac{\hbar^2}{2m}\left[\underbrace{\psi'+\psi'}_{2\psi'} + x\cancel{\psi''}\right] + \frac{\hbar^2}{2m}\cancel{x}\psi''$$

$$= -\frac{\hbar^2}{m}\psi' = i\frac{\hbar}{m}\left(-i\hbar\frac{d}{dx}\right)\psi$$

$$= i\frac{\hbar}{m}p\psi$$

$$[a, a^\dagger] = a a^\dagger - a^\dagger a$$

$$a^\dagger a = \frac{1}{\hbar\omega} H - 1/2$$

$$a = \frac{1}{\sqrt{2}} \sqrt{\alpha} \left(x + \frac{i}{m\omega} p \right) \qquad a^\dagger = \frac{1}{\sqrt{2}} \sqrt{\alpha} \left(x - \frac{i}{m\omega} p \right)$$

$$a a^\dagger = \frac{1}{2} \alpha \left(x + \frac{i}{m\omega} p \right) \left(x - \frac{i}{m\omega} p \right)$$

$$= \frac{1}{2} \alpha \left[x^2 - \frac{i}{m\omega} [x, p] + \frac{1}{m^2\omega^2} p^2 \right]$$

$$\frac{1}{2} \frac{m\omega}{\hbar} \left[x^2 + \frac{1}{m^2\omega^2} p^2 + \frac{\hbar}{m\omega} \right]$$

$$\frac{1}{\hbar\omega} \left[\frac{1}{2} m\omega^2 x^2 + \frac{1}{2} \frac{p^2}{m} \right] + \frac{1}{2}$$

$$= \frac{1}{\hbar\omega} H + \frac{1}{2}$$

So $\quad a a^\dagger - a^\dagger a = \left[\frac{1}{\hbar\omega} H + \frac{1}{2} \right] - \left[\frac{1}{\hbar\omega} H - \frac{1}{2} \right]$

$$= 1$$
