MATH348-Advanced Engineering Mathematics		**Homework: Fourier Series**
	INTEGRATION REVIEW AND INTRODUCTION TO FOURIER SERIES	
Text: 11.1-11.2	Lecture Notes : 9	Lecture Slides: N/A

Quote of Homework: Fourier Series - Solutions	
And of course Henry the horse dances the waltz!	
	The Beatles : Being for the Benefit of Mr. Kite! (1967)

1. Integration Review

1.1. **Integration by Parts.** $\int x^3\cos(5x)dx$ Integration by parts is common when working with Fourier series and can efficiently be done through tables. If you have never done this then wikipedia has a good *article*. The result is,

u	dv
x^3	$\cos(5x)$
$3x^2$	$\frac{\sin(5x)}{5}$
$6x$	$\frac{-\cos(5x)}{25}$
6	$\frac{-\sin(5x)}{125}$
0	$\frac{\cos(5x)}{625}$

$$\int x^3\cos(5x)dx = \frac{x^3\cdot\sin(5x)}{5}+\frac{3x^2\cdot\cos(5x)}{25}-\frac{6x\cdot\sin(5x)}{125}-\frac{6\cdot\cos(5x)}{625}+c$$

1.2. **Integration by ?** $\int x^2\sin(2x^3)dx$ Don't let the power-term fool you. This integration is done via substitution.

$$\begin{aligned}\int x^2\sin(2x^3)dx &= \frac{1}{6}\int\sin(u)du,\ {\scriptstyle u=2x^3 \atop du=6x^2}\\ &= \frac{1}{6}(-\cos(u))+c=\frac{-\cos(2x^3)}{6}+c\end{aligned}$$

1.3. **Tricky IBP or Tricky Algebra.** $\int e^{ax}\cos(bx)dx$ and $\int e^{ax}\sin(bx)dx$ Both of these integrals require a cyclic integration by parts argument, which can be found *here*. It is easier to avoid the integration by parts altogether. Consider,

$$\int e^{ax}e^{ibx}dx=\int e^{(a+ib)x}dx \tag{1}$$

$$=\frac{1}{a+bi}e^{(a+bi)x} \tag{2}$$

$$=e^{ax}\left(\frac{a}{a^2+b^2}-i\frac{b}{a^2+b^2}\right)e^{ibx} \tag{3}$$

$$=\frac{a\cos(bx)+b\sin(bx)}{a^2+b^2}e^{ax}+i\frac{a\sin(bx)-b\cos(bx)}{a^2+b^2}e^{ax}, \tag{4}$$

which implies that

$$\mathrm{Re}\left(\int e^{ax}e^{ibx}dx\right)=\int e^{ax}\cos(bx)dx=\frac{a\cos(bx)+b\sin(bx)}{a^2+b^2}e^{ax} \tag{5}$$

$$\mathrm{Im}\left(\int e^{ax}e^{ibx}dx\right)=\int e^{ax}\sin(bx)dx=\frac{a\sin(bx)-b\cos(bx)}{a^2+b^2}e^{ax} \tag{6}$$

1.4. **Integration of *Delta 'Functions'*.** Justify that $\int_{-\infty}^{\infty}\delta(x-x_0)g(x)dx=g(x_0)$ for some $x_0\in\mathbb{R}$

Recall that the working definition of a Dirac delta 'function' is,

$$\delta(x-x_0)=0,\ \text{for all } x\neq x_0, \tag{7}$$

such that

$$\int_{x_0-\epsilon}^{x_0+\epsilon}\delta(x-x_0)dx=1,\ \text{for all } \epsilon>0. \tag{8}$$

It is disheartening to know that no function can do this but rest assured that the use of this replacement rule is made rigorous in the *theory of generalized functions* or so-called *distributions*. These functionals where in use long before they were made rigorous and this is what you need to know,

$$\int_{-\infty}^{\infty} \delta(x-x_0)g(x)dx = \int_{-\infty}^{\infty} \delta(x-x_0)g(x_0)dx \tag{9}$$

$$= g(x_0)\int_{-\infty}^{\infty} \delta(x-x_0)dx \tag{10}$$

$$= g(x_0). \tag{11}$$

We think about this rule as a replacement rule, which says that if you integrate function against the delta functional then you evaluate the function at the point where the delta functional is not zero.

1.5. **Integrals of *Gaussian* Functions.** Show that $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$ Show that $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$

This integral is one of the more important integrals from physics and probability and is the function associated with the so-called 'bell-curve.' We would like to know about the area under its curve but alas we do not know its anti-derivative.[1] Consider defining $I = \int_{-\infty}^{\infty} e^{-x^2}dx$ then,

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2}dx \int_{-\infty}^{\infty} e^{-y^2}dy \tag{12}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}dxdy \tag{13}$$

$$= \int_{0}^{2\pi}\int_{0}^{\infty} re^{-r^2}drd\theta \tag{14}$$

$$= \int_{0}^{2\pi}\int_{0}^{\infty} \frac{e^{-u}}{2}dud\theta \tag{15}$$

$$= \int_{0}^{2\pi} \frac{1}{2}d\theta \tag{16}$$

$$= \pi, \tag{17}$$

implies that $I = \sqrt{\pi}$. Though it functionally works, this argument is imprecise. Notice that the domain of integration changes with the coordinate change from (13)-(14) and consequently so does the geometry. These integrals are improper and should be thought of as definite integrals whose limits of integrations are themselves limits. So, the thinking is like this, at (13) you are integrating over a square where the lengths of the sides of the square are undergoing a limiting process, which makes them infinitely long. However, at (14) the geometry is a circle whose radius is undergoing a limiting process making it infinitely long. Why should the integral over the circle converge to the same value over the square? Well, they consume all of space but that isn't a really good reason.

Consider now, bounding the square in Cartesian below and above by a circles and taking the limit as the lengths and radii go to infinity. In this limit the integration on the circles will give you $I = \pi$ and as this process is going on the square will be *squeezed* above and below by these circles and must naturally converge to the same number. Thank you squeeze theorem.

1.6. **Orthogonality.** Show that $\int_{0}^{2\pi} \sin(nx)\cos(mx)dx = 0$ for all $n, m \in \mathbb{N}$

[1] One could power-expand the integrand and since its Taylor series is absolutely convergent one could exchange integration with the summation and conduct term-wise integration. However, this approach is not helpful since you will be left with some infinite series, which you will then have to sum.

First, this can be done with the trigonometric identity $\sin(nx)\cos(mx) = \frac{1}{2}\left[\sin(nx-mx)+\sin(nx-mx)\right]$. We will use the following identities instead, $2\cos(x) = e^{ix}+e^{-ix}$, $2i\sin(x) = e^{ix}-e^{-ix}$ and $e^{\pm in\pi} = (-1)^n$, where n is an integer. Now we have,

$$\int_0^{2\pi} \sin(nx)\cos(mx)dx = \int_0^{2\pi} \left[\frac{e^{inx}-e^{-inx}}{2i}\right]\left[\frac{e^{imx}+e^{-imx}}{2}\right]dx \tag{18}$$

$$= \int_{-\pi}^{\pi} \left[\frac{e^{in(u+\pi)}-e^{-in(u+\pi)}}{2i}\right]\left[\frac{e^{im(u+\pi)}+e^{-im(u+\pi)}}{2}\right]du \tag{19}$$

$$= \int_{-\pi}^{\pi} (-1)^n(-1)^n \left[\frac{e^{inu}-e^{-inu}}{2i}\right]\left[\frac{e^{imu}+e^{-imu}}{2}\right]du \tag{20}$$

$$= \int_{-\pi}^{\pi} \sin(nu)\cos(mu)du \tag{21}$$

$$= 0, \tag{22}$$

since the integrand is odd and the domain of integration is $(-\pi,\pi)$

1.7. **More Orthogonality.** Show that $\int_a^b e^{i\frac{n\pi}{L}x}e^{-i\frac{m\pi}{L}x}dx = 2L\delta_{mn}$ where $L = \frac{b-a}{2}$ and for all $n,m \in \mathbb{Z}$. This integral is important for generalizing Fourier series to any finite domain of $\mathbb{R}$. First the integral,

$$\int_a^b e^{i\frac{n\pi}{L}x}e^{-i\frac{m\pi}{L}x}dx = \int_a^b e^{\frac{i}{\pi L}(n-m)x}dx \tag{23}$$

$$= e^a \int_0^{b-a} e^{\frac{i\pi}{L}(n-m)u}du \tag{24}$$

$$= e^a \frac{L}{i\pi(n-m)} \left. e^{\frac{i\pi}{L}(n-m)u}\right|_0^{b-a} \tag{25}$$

$$= e^a \frac{L}{i\pi(n-m)}\left(e^{\frac{i\pi}{L}(n-m)(b-a)}-1\right) \tag{26}$$

$$= e^a \frac{L}{i\pi(n-m)}\left(e^{2\pi i(n-m)}-1\right) \tag{27}$$

$$= 0, \text{ for } n \neq m, \tag{28}$$

provides orthogonality. While noting for $n = m$,

$$\int_a^b e^{i\frac{n\pi}{L}x}e^{-i\frac{m\pi}{L}x}dx = \int_a^b e^0 dx \tag{29}$$

$$= b-a \tag{30}$$

$$= 2L, \tag{31}$$

provides the square of the vectors length. This result indicates that orthogonality is maintained for functions that are defined off of the symmetric interval $(-\pi,\pi)$, which are 2L-periodic.

2. Introduction to Fourier Series

2.1. **Wikipedia.** Go to `http://en.wikipedia.org/wiki/Fourier_series` and read the introductory material on Fourier Series and describe in your own words the purpose and application of Fourier Series.

A Fourier series is a method of decomposing periodic functions in terms of sines and cosines with discrete frequencies. Typically, a Fourier series is used to understand a function's frequency spectrum and because of this, appears heavily in signal analysis. However, as a tool, it is very powerful and appears in many applications having to do with PDE's. This is due to the fact that every reasonable function defined on a finite spatial domain has access to a Fourier decomposition. Since most physical problems modelled by PDE occur on a finite spatial domains Fourier series appear quite naturally when finding solutions through separation of variables.

2.2. **Graphing.** Using the Java Applet found at `http://www.sunsite.ubc.ca/LivingMathematics/V001N01/UBCExamples/Fourier/fourier.html`, use the applet to graph a truncated Fourier Series approximating the saw-tooth function. What occurs at the points jump-discontinuity?

Near the points of discontinuity a truncated Fourier series will display a ringing/oscillations, or what is called Gibb's phenomenon, which is an consequence of a linear combination of continuous functions trying to approximate a jump discontinuity. This error can be minimized through the use of low-pass filters or wavelet transforms using the so-called Harr basis. If the Fourier series is not truncated then at the point of discontinuity, the Fourier series will average the left-hand and right-hand limits of the function and Gibb's phenomenon will stop.

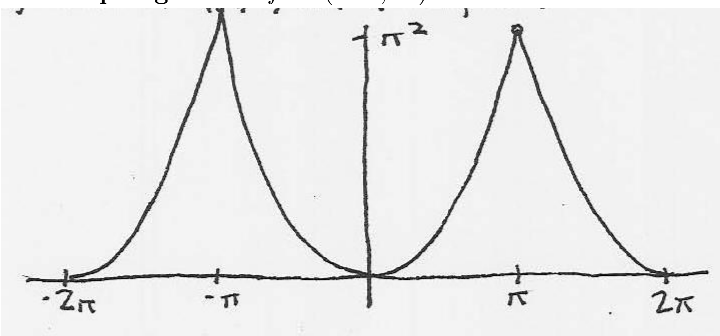
2.3. **Truncated Fourier Series.** Read, as much as you can, of `http://en.wikipedia.org/wiki/Gibbs_phenomenon`. The sum of a finite, or infinite amount of periodic functions is periodic. Is this always true for both finite and infinite sums of continuous functions? Can you think of a counterexample? [2]

The sum of periodic functions is always periodic. However, the infinite sum of continuous functions may be discontinuous. The square-wave is an example of Fourier series, which is the infinite sum of continuous functions, that has jump discontinuities.

3. Fourier Series : Even

Let $f(x) = x^2$ for $x \in (-\pi, \pi)$ be such that $f(x + 2\pi) = f(x)$.

3.1. **Graphing.** Sketch f on $(-2\pi, 2\pi)$.



3.2. **Symmetry.** Is the function even, odd or neither?
The function is even. This can be seen by the graph above, which is symmetric about the y-axis. Also, $f(-x) = (-x)^2 = x^2 = f(x)$.

3.3. **Integrations.** Determine the Fourier coefficients a_0, a_n, b_n of f.

$$
\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx \\
&= \left.\frac{x^3}{3\pi}\right|_0^{\pi} = \frac{\pi^2}{3}, \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos(nx)dx = \frac{2}{\pi}\int_0^{\pi} x^2\cos(nx)dx \\
&= \frac{2}{\pi}\left[\frac{x^2}{n}\sin(nx) + \frac{2x}{n^2}\cos(nx) - \frac{2}{n^3}\sin(nx)\right]_0^{\pi} \\
&= \frac{2}{\pi}\left[\frac{2\pi}{n^2}\cos(n\pi)\right] = \frac{4(-1)^n}{n^2}, \\
b_n &= 0, \\
f(x) &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos(nx)
\end{aligned}
$$

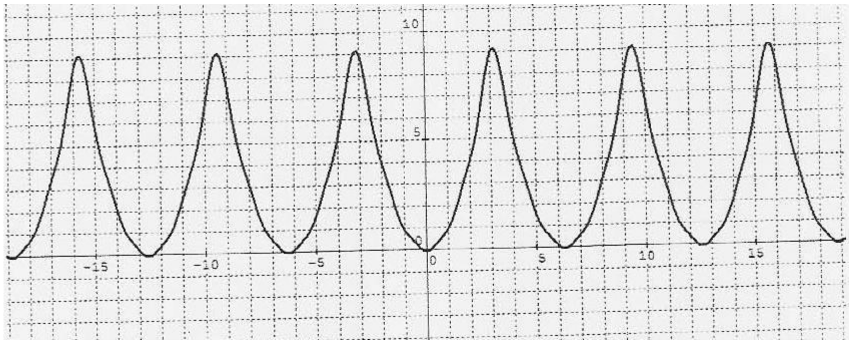
$$a_0 = \frac{\pi^2}{3}, \quad a_n = \frac{4(-1)^n}{n^2}, \quad b_n = 0$$

[2]These questions are meant to lead you. Remembering that sine and cosine are examples of continuous periodic functions, you should be thinking about the following string of thoughts.

(1) Fourier series represent an 'arbitrary' periodic function in terms of known periodic functions.
(2) Increasing the number of terms in a Fourier series creates better and better sinusoidal wave-form fits of the function f and in the limit of infinitely many terms this fit is exact 'almost-everywhere'.
(3) Hopefully by the time you do this problem we would have mentioned in class that the Fourier series representation of a function converges in the sense of averages and that since jump-discontinuities are integrable-discontinuities the Fourier series would average the right and left hand limits of the function at the point of discontinuity. This will happen indifferent to the actual value of the function at the point of discontinuity. Thus the Fourier series may actually differ from its function at the boundaries of its periodic-domains! In this way we take = to mean equality *almost everywhere* (`http://en.wikipedia.org/wiki/Almost_everywhere`).

So, we have that the sawtooth example from class and the square-wave example online are examples where the infinite sum of continuous periodic functions converges to a periodic function with jump-discontinuities.

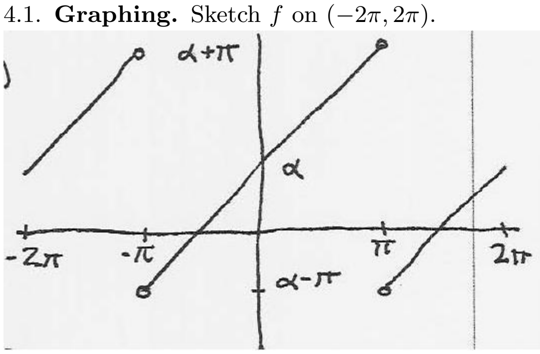
3.4. **Truncation.** Using `http://www.tutor-homework.com/grapher.html`, or any other graphing tool, graph the first five terms of your Fourier Series Representation of f.



4. Fourier Series : Oddish

Let $f(x) = x + \alpha$ for $x \in (-\pi, \pi)$ and $\alpha \in \mathbb{R}$ be such that $f(x + 2\pi) = f(x)$.

4.1. **Graphing.** Sketch f on $(-2\pi, 2\pi)$.



4.2. **Symmetry.** Is the function even, odd or neither?
If $\alpha = 0$ then the function is odd. However, if $\alpha \neq 0$ then the function is the sum of an even function with an odd function and is, consequently, neither even nor odd.

4.3. **Integrations.** Determine the Fourier coefficients a_0, a_n, b_n of f.

We have that the Fourier Series of f(x) should be the addition of the Fourier series for $f_1(x) = x$ and the Fourier Series for $f_2(x) = \alpha$.

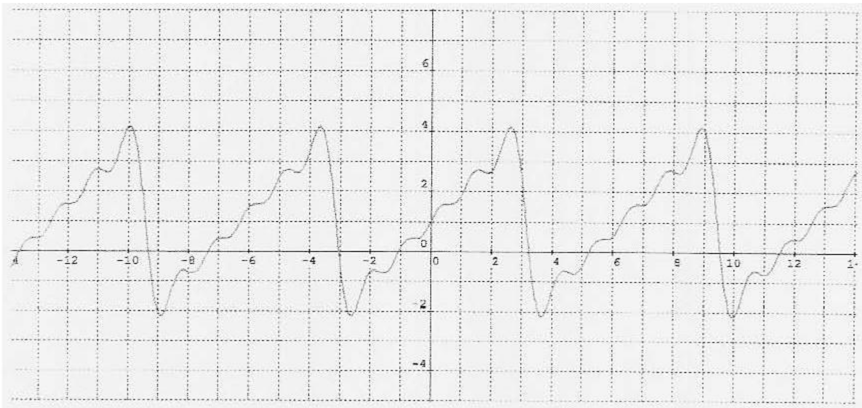
We have from class that

$$f_1(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nx)$$

Formulas from Kreysig p. 480.

$$\begin{aligned}
&\text{For } f_2(x): \\
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \alpha dx = \frac{\alpha x}{2\pi}\Big|_{-\pi}^{\pi} = \alpha \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \alpha \cos(nx) dx = \frac{\alpha}{\pi}\sin(nx)\Big|_{-\pi}^{\pi} = 0 \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \alpha \sin(nx) dx = \frac{\alpha}{\pi}\cos(nx)\Big|_{-\pi}^{\pi} = 0 \\
\text{Thus,} &\quad \text{for } f(x) = x + \alpha \\
f(x) &= \alpha + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n}\sin(nx) \\
&\quad a_0 = \alpha, \quad a_n = 0, \quad b_n = \frac{2(-1)^{n+1}}{n}
\end{aligned}$$

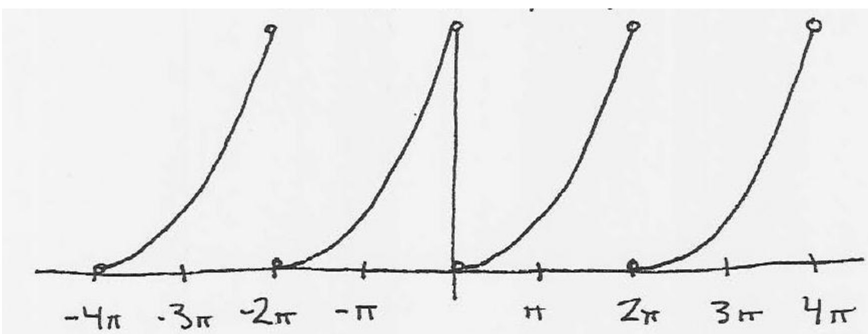
4.4. **Truncation.** Using `http://www.tutor-homework.com/grapher.html`, or any other graphing tool, graph the first five terms of your Fourier Series Representation of f assuming that $\alpha = 1$.



5. Fourier Series : Nonstandard Domain

Let $f(x) = x^2$ for $x \in (0, 2\pi)$ be such that $f(x + 2\pi) = f(x)$.

5.1. **Graphing.** Sketch f on $(-4\pi, 4\pi)$.



5.2. **Symmetry.** Is the function even, odd or neither?

Don't let the quadratic function fool you. This function is neither even nor odd as can be seen by the previous graph.

5.3. **Integrations.** Determine the Fourier coefficients a_0, a_n, b_n of f.

This problem highlights an important property of Fourier series. The idea is this. If you want to create a Fourier series for the graph that is given and you want to use the standard integrations on $(-\pi, \pi)$ then you must use the piecewise definition of f given by,

$$f(x) = \begin{cases} (x + 2\pi)^2, & -\pi < x \le 0 \\ x^2, & 0 < x < \pi \end{cases} . \tag{32}$$

This is the same as the graph above and defined on $(-\pi, \pi)$ so that it is ready for use with the standard formulae. However, this calculation will be cumbersome. Instead note the following logic,

(1) A Fourier series is the linear combination of sine/cosine basis vectors that also obey an orthogonality condition.
(2) From this orthogonality condition the coefficients in the linear combination can be found in terms of integrals.
(3) The orthogonality condition was found with the inner-product $< f, g >= \int_{-\pi}^{\pi} f(x)g(x)dx$.
(4) The integral defined in 1.7 of this homework obeys all the rules of an inner-product and shows that the imaginary-exponential functions are orthogonal on any $2L-$domain.
(5) Thus the sine/cosine functions are orthogonal on any $2L-$domain. Consequently, the Fourier coefficients can be re-derived for any $2L-$domain.

In this case the following integrals are taken on the principle domain of the periodic function and b/c of this the integrations are simpler.

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_0^{2\pi} x^2 dx = \left.\frac{x^3}{6\pi}\right|_0^{2\pi} = \frac{4\pi^2}{3} \\
a_n &= \frac{1}{\pi}\int_0^{2\pi} x^2 \cos(nx) dx = \\
&= \frac{1}{\pi}\left[\frac{x^2}{n}\sin(nx) + \frac{2x}{n^2}\cos(nx) - \frac{2}{n^3}\sin(nx)\right]_0^{2\pi} \\
&= \frac{1}{\pi}\left[\frac{4\pi}{n^2}\right] = \frac{4}{n^2} \\
b_n &= \frac{1}{\pi}\int_0^{2\pi} x^2 \sin(nx) dx \\
&= \frac{1}{\pi}\left[\frac{-x^2}{n}\cos(nx) + \frac{2x}{n^2}\sin(nx) + \frac{2}{n^3}\cos(nx)\right]_0^{2\pi} \\
&= \frac{1}{\pi}\left[\frac{-4\pi^2}{n} + \frac{2}{n^3} - \frac{2}{n^3}\right] = \frac{-4\pi}{n} \\
f(x) &= \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\left[\frac{4}{n^2}\cos(nx) - \frac{4\pi}{n}\sin(nx)\right]
\end{aligned}$$

$$a_0 = \frac{4\pi^2}{3} \quad a_n = \frac{4}{n^2} \quad b_n = \frac{-4\pi}{n}$$

5.4. **Truncation.** Using `http://www.tutor-homework.com/grapher.html`, or any other graphing tool, graph the first five terms of your Fourier Series Representation of f.

Key Point: The role of the Fourier coefficients is to take in periodic data, say $f(x)$, on some domain say, $(0, 2\pi)$. The role of the Fourier series is to then take the data supplied by the coefficients and repeat this graph to the right and to the left of the principle domain. NICE