PHGN361 2010 Exam 2: NAME

Start from fundamental principles and derive all results. **Explain each step for credit.**

1. (12 points) A solid conducting sphere is held at constant voltage V_0. **Using separation of variables, derive in detail,** using equations and words, the voltage inside the sphere. The radial solution to Laplace's equation in spherical coordinates is $R(r) = A_l r^l + \frac{B_l}{r^{l+1}}$ while the angular solution is $P_l(\cos\theta)$, where l is an integer and A_l and B_l are constants. Apply the integral $\int_{-1}^{1} P_l(x)P_m(x)dx = \frac{2}{2l+1}\delta_{lm}$ if appropriate while you need not evaluate other integrals.

You may start from $\Theta(\theta)$

$\nabla^2 V = 0$ in spherical coords leads to $R(r)\,\Theta(\theta)$. However this does not lead to a solution which satisfies general boundary conditions. To do this use the superposition principle & orthogonality of the $P_l(\cos\theta)$: $V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$. Inside the sphere $r=0$ V cannot go to ∞ so $B_l = 0$. Now apply this soln at the boundary. $V(r=R,\theta) = \sum_l A_l R^l P_l(\cos\theta) = V_0$

Multiply both sides by $P_m(\cos\theta)$ & integrate to use orthogonality.

$$\int_{-1}^{1}\sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta)P_m(\cos\theta)\,d(\cos\theta) = \int_{-1}^{1} V_0 P_m(\cos\theta)\,d(\cos\theta)$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\sum_{l=0}^{\infty} A_l R^l \frac{2}{2m+1}\delta_{lm} = A_m R^m \frac{2}{2m+1} = V_0 \int_{-1}^{1} P_m(\cos\theta)\,d(\cos\theta)$$

Solve for $A_m = \frac{2m+1}{2R^m} V_0 \int_{-1}^{1} P_m(\underbrace{\cos\theta}_{x})\,d(\underbrace{\cos\theta}_{x})$

$$V = \sum_l A_l r^l P_l = A_0 R^0 = \frac{2R^0}{2R^0}V_0 = V_0$$

(3 points) What answer do you expect and why do you expect it?

E inside conductor $= 0 \Rightarrow -\int \vec{E}\cdot d\vec{l} = 0 = \Delta V$ so V must be constant, note that $\int_{-1}^{1} P_m(x)\, \underset{P_0(x)}{1}\, dx$ (inside path)

$$= \int_{-1}^{1} P_m(x)P_0(x)\,dx = \frac{2}{2m+1}\delta_{m0} = 2 \Rightarrow \text{only } A_0 \text{ survives in sum}$$