# Traveling waves

Note Title 2/8/2008

$$\frac{\partial^2 f}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = 0 \qquad \text{1D wave eqn.}$$

Let $f = f(x \pm ct)$ or $f(kx \pm \omega t)$

$$\frac{\partial f}{\partial x} = k f' \qquad \frac{\partial^2 f}{\partial x^2} = k^2 f''$$

$$\frac{\partial f}{\partial t} = \pm \omega f' \qquad \frac{\partial^2 f}{\partial t^2} = \omega^2 f''$$

plug into wave eqn.

$$k^2 f'' - \frac{\omega^2}{c^2} f'' = 0 = \left(k^2 - \frac{\omega^2}{c^2}\right) f'' = 0$$

true if $k^2 = \frac{\omega^2}{c^2}$, no matter what $f$.

the shape $f$ moves with "speed" $c$ to the left or right

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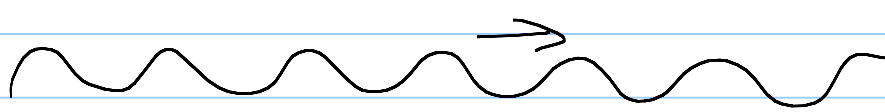
Eg. $f(x,t) = A \sin(kx - \omega t)$



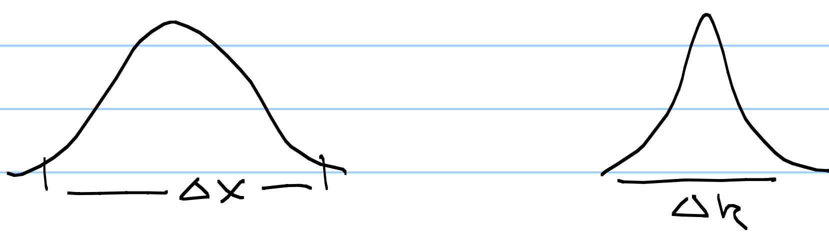
Super position

$$f(x,t) = \sum_i A_i \sin(k_i x - \omega_i t)$$

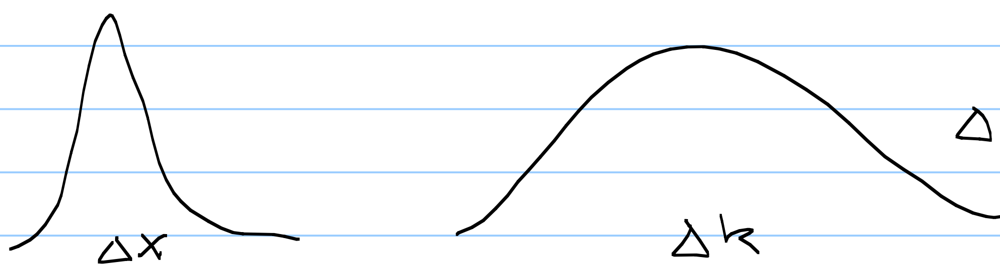
or

$$\int A(k) \sin(kx - \omega t)\, dk$$

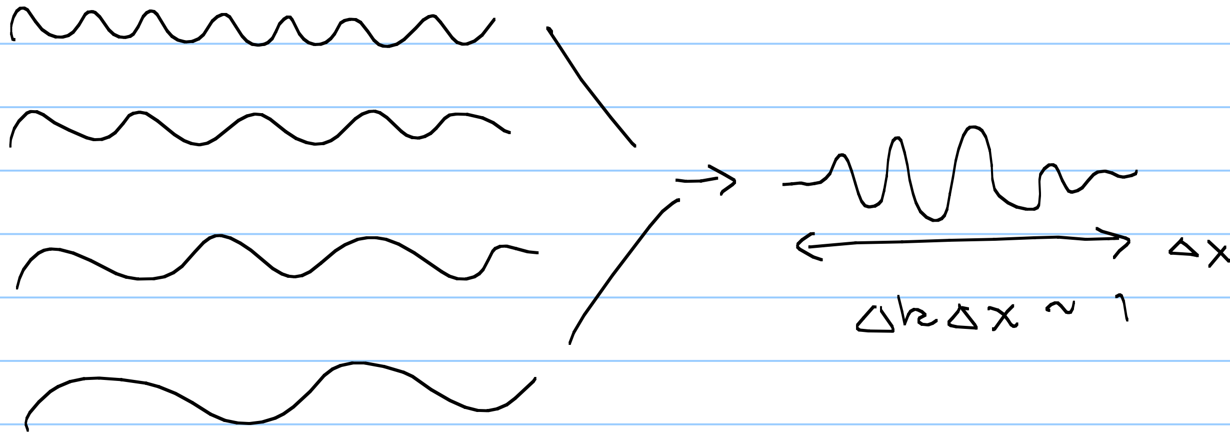
to produce a localized wave packet requires lots of wave numbers




$\Delta x \Delta k \sim 1$




$\Delta x \Delta k \sim 1$




$\Delta k \Delta x \sim 1$

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Note Title 2/8/2008

Free particle $V(x) = 0$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

$$\psi'' + \frac{2mE}{\hbar^2}\psi = 0$$

$$\psi'' + k^2\psi = 0 \qquad k^2 = \frac{2mE}{\hbar^2}$$

$$\psi(x) = Ae^{ikx} + Be^{-ikx}$$

$$\Psi(x,t) = Ae^{i(kx - Et/\hbar)} + Be^{-(kx + Et/\hbar)}$$

factor out $a\,k$ $\quad e^{ik(x - \frac{E}{kh}t)}$

$$\frac{E}{k\hbar} = \frac{\hbar k}{2m}$$

$$\Psi(x,t) = A\underbrace{e^{ik(x - \frac{k\hbar}{2m}t)}} + B\underbrace{e^{-ik(x - \frac{k\hbar}{2mE})}}$$

plane waves moving to left + right

$$x \pm vt$$

$$v = \frac{\hbar k}{2m}$$

$$k = \frac{\pm\sqrt{2mE}}{\hbar}$$

$k > 0$ traveling to right

$< 0$ left

$$\boxed{\text{deBroglie} \qquad p = \hbar k}$$

$$v = \frac{\hbar k}{2m} = \frac{p}{2m} = \frac{1}{2} v_{classical}$$

Not normalizable. form a Linear combination

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k)\, e^{i(kx - \hbar k^2/2m\, t)}\, dk$$

for appropriate choice of $\phi$ this can be normalized

Now, as usual, we will be given $\Psi(x, t=0)$ and asked to find $\Psi(x,t)$.

$$\Psi(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k)\, e^{ikx}\, dk$$

Hence $$\boxed{\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x,0)\, e^{-ikx}\, dx}$$

Recall from 311

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) e^{ikx}\, dk \Leftrightarrow F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$$

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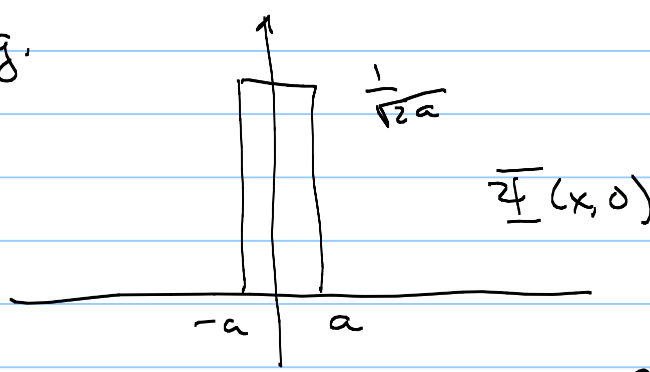
Eg.




$$\phi(k) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2a}} \underbrace{\int_{-a}^{a} e^{-ikx}\, dx}_{\frac{1}{-ik} e^{-ikx} \Big|_{-a}^{a}}$$

$$\frac{1}{2}\ \frac{1}{\sqrt{\pi a}}\ \frac{i}{k}\ \underbrace{\left[ e^{-ika} - e^{+ika} \right]}_{-2i\sin ka}$$

$$= \frac{1}{\sqrt{\pi a}}\ \frac{\sin ka}{k}$$

So

$$\psi(x,t) = \frac{1}{\pi}\ \frac{1}{\sqrt{2a}} \underbrace{\int_{-\infty}^{\infty} \frac{\sin ka}{k}\ e^{-ik\left(x - \frac{\hbar k}{2m} t\right)} dk}_{\text{can't be don analytically}}$$

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Standard form of a wave packet

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k)\, e^{i(kx - \omega t)}$$

in our case $\omega = \hbar k^2/2m$

$\omega(k)$ is called the dispersion relation

$$\frac{\omega}{k} = v_{phase} \qquad \frac{d\omega}{dk} = v_{group}$$

$\omega = \frac{\hbar k^2}{2m}$

for free particle $\frac{d\omega}{dk} = \frac{k\hbar}{m}$

$\frac{\omega}{k} = \frac{\hbar k}{2m}$

$V_{group} = V_{classical} = 2 V_{phase}$