Note Title 11/1/2006

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Some basic theorems:

Let $\phi[f] = F$ denote the Fourier transform operator. Then

$$\phi[c_1 f_1 + c_2 f_2] = c_1 \phi[f_1] + c_2 \phi[f_2]$$

Let $f^{(k)}(x) = \dfrac{d^k f}{dx^k}$ then

$$\phi[f^{(N)}] = (ik)^n \phi[f]$$

E.g. $\phi[f^{(1)}] = \phi\left[\dfrac{df}{dx}\right]$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{df}{dx} e^{-ikx} \, dx$$

integrate by parts

$$= \frac{1}{\sqrt{2\pi}} \left[\underbrace{f e^{-ikx} \Big|_{-\infty}^{\infty}}_{?} + ik \underbrace{\int_{-\infty}^{\infty} f e^{-ikx} \, dx}_{\phi[f]} \right]$$

we assume that the endpoints are zero.

$$\phi\left[\frac{df}{dx}\right] = ik\, \phi[f]$$

etc.

You can do this over and over to get the result

$$\phi[f^{(N)}] = (ik)^N \phi[f]$$

shift theorem

$$\phi[f(t-c)] = e^{-i\omega c}\,\phi[f]$$

$$\equiv \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t-c)\,e^{-i\omega t}\,dt$$

$$t-c=\tau$$
$$t=\tau+c$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\tau)\,e^{-i\omega\tau}\,e^{-i\omega c}\,d\tau$$

$$= e^{-i\omega c}\underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\tau)\,e^{-i\omega\tau}\,d\tau}_{\phi[f]} \checkmark$$

Similarly

$$\phi[e^{ict} f(t)] = F(\omega-c)$$

Phase shift in 1 domain corresponds to displacement in the other.

Convolution

$f(t)$, $g(t)$ 2 functions

$\downarrow$ $\downarrow$ their

$F(\omega)$ $G(\omega)$ Fourier trans

$$F(\omega)G(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt \int_{-\infty}^{\infty} g(t)\, e^{-i\omega t}\, dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, g(t')\, e^{-i\omega t'}\, dt'\, dt$$

$$= \frac{1}{2\pi}\iint_{-\infty}^{\infty} f(t)\, g(t')\, e^{-i\omega(t+t')}\, dt'\, dt$$

$$t + t' = \tau$$

can bump either t or t'

in favor of τ:

$t' = \tau - t$

$$\frac{1}{2\pi} \int\int_{-\infty}^{\infty} f(t)\, g(\tau - t)\, e^{-i\omega\tau}\, d\tau\, dt$$

$$\int_{-\infty}^{\infty} f(t)\, g(\tau - t)\, dt$$

$$\equiv [f * g](\tau)$$

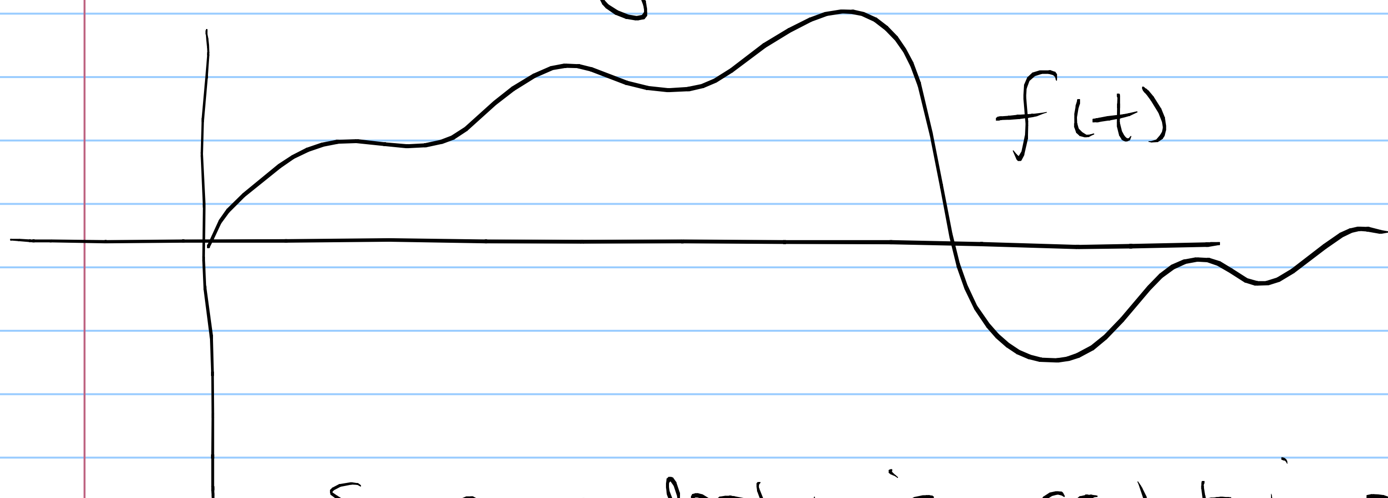
$$F(\omega)\, G(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega\tau}\, [f*g](\tau)\, d\tau$$

or

$$[f * g](\tau) = \int_{-\infty}^{\infty} F(\omega)\, G(\omega)\, e^{i\omega\tau}\, d\omega$$

convolution theorem

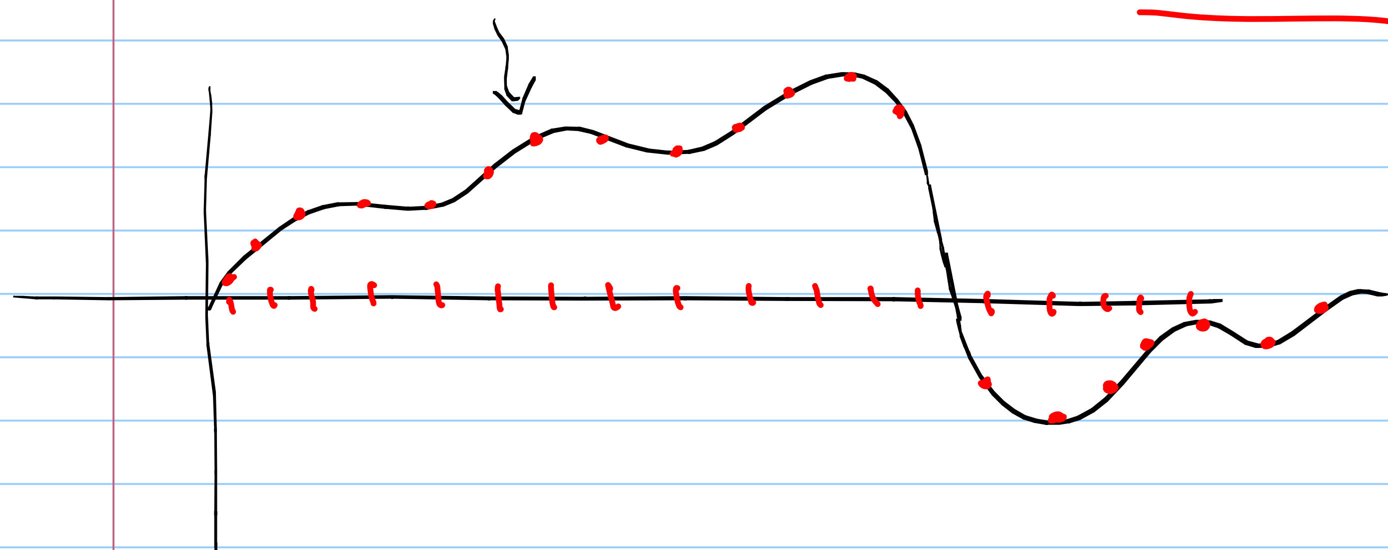
Sampling



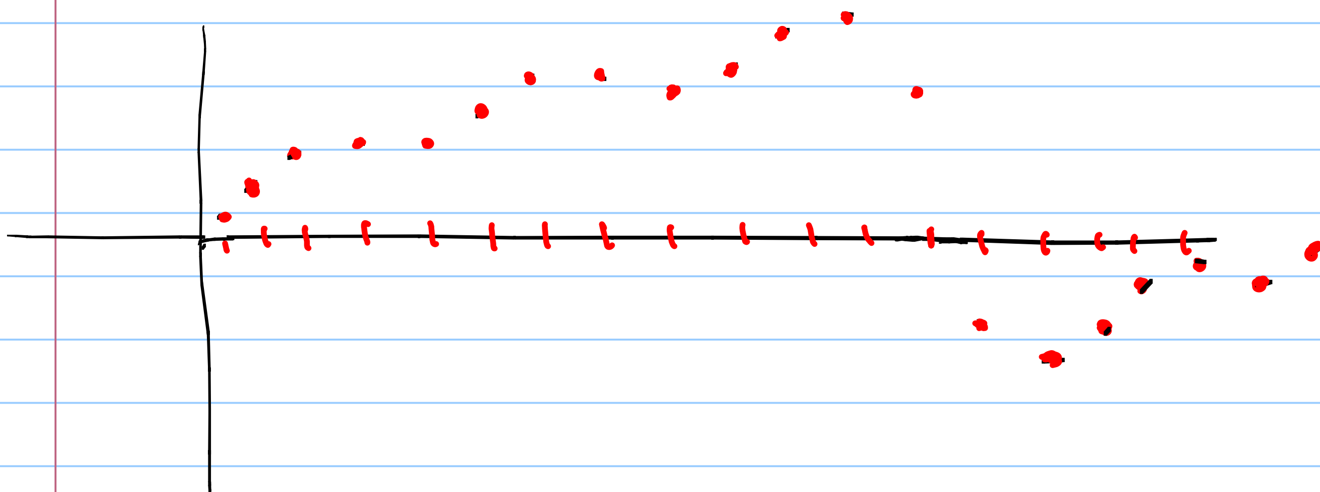
Some underlying continuous function, unknown to us.

$(t_i, y_i) = f(t_i)$

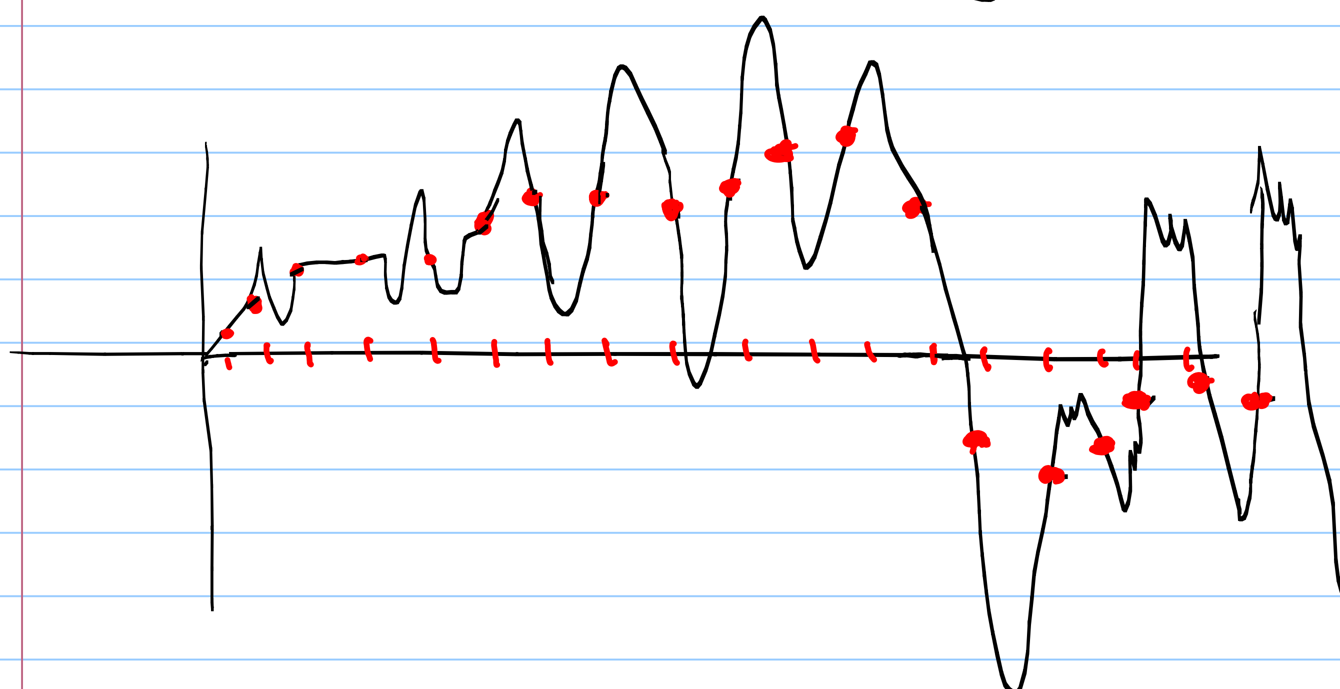
No noise or errors



This is what we actually measure
not f(t)



But what's to prevent
f(t) from looking like: ?



No instrument can measure 0 frequency or ∞ frequency.

All real devices are

Band limited

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-2\pi f_c}^{2\pi f_c} F(\omega) e^{-i\omega t} \, d\omega$$

i.e. $F(\omega)$ is nonzero only on the interval

$$[-2\pi f_c, \ 2\pi f_c]$$

Hence we represent $F(\omega)$ as a Fourier Series

$$F(\omega) = \sum_{N=-\infty}^{\infty} \phi_N \, e^{i\omega N/2f_s}$$

$$\phi_N = \frac{1}{\sqrt{2\pi}\, 2f_s} \int_{-2\pi f_s}^{2\pi f_c} F(\omega)\, e^{-i\omega N/2f_c} \, d\omega$$

compare this with

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-2\pi f_c}^{2\pi f_c} F(\omega) e^{-i\omega t} \, d\omega$$

evaluate t at $\frac{N}{2f_c}$

$$\boxed{\phi_N = \frac{f(N/2f_c)}{2f_c}}$$

These are the Fourier coefficients of $F(\omega)$, the continuous but band-limited function

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-2\pi f_c}^{2\pi f_c} e^{-i\omega t} \left[\sum_{N=-\infty}^{\infty} \frac{f(N/2f_c)}{2f_c} e^{i\omega N/2f_c} \right] d\omega$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{N=-\infty}^{\infty} \frac{f(N/2f_c)}{2f_c} \int_{-2\pi f_c}^{2\pi f_c} e^{i\omega\left(\frac{N}{2f_c} - t\right)} d\omega$$

$$\frac{1}{i\left(\frac{N}{2f_c} - t\right)} \; e^{i\omega(\quad)} \Big|_{-2\pi f_c}^{2\pi f_c}$$

$$\vdots$$

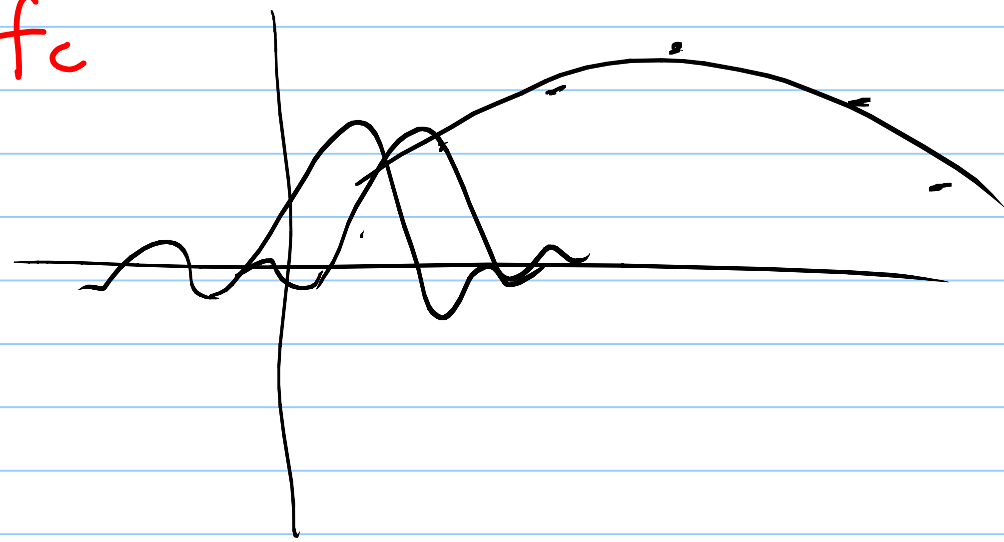
$$\frac{2 \sin(\pi(2f_c t - N))}{\pi(2f_c t - N)}$$

put it all together

$$\boxed{f(t) = \sum_{N=-\infty}^{\infty} f(N/2f_c) \frac{\sin(\pi(2f_c t - N))}{\pi(2f_c t - N)}}$$

A Band-limited continuous function is completely captured by discrete samples taken every

$\frac{1}{2f_c}$



$$(f,g) \equiv \int_{-L}^{L} f^*(x)\, g(x)\, dx$$

$$(f,f) = \int_{-L}^{L} \underbrace{f^*(x)\, f(x)}_{|f(x)|^2}\, dx$$

$$f(x) = \int_{-\infty}^{\infty} f(x')\, \underbrace{K(x'-x)}_{\delta(x'-x')}\, dx$$