orthogonal projection

Note Title 10/19/2007

finite dimensional

$z = a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3 + \dots a_N\hat{e}_N$

infinite dimensional

$$f(x) = \frac{a_0}{2} \sum_{N=1}^{\infty} a_N \sin\left(\frac{N\pi x}{\ell}\right) + b_N \cos\left(\frac{N\pi x}{L}\right)$$

$$a_i = \frac{\hat{e}_i \cdot z}{\hat{e}_i \cdot \hat{e}_i}$$ works because $\hat{e}_i \cdot \hat{e}_j = 0$ if $i \neq j$

what's the corresponding result for sins & cosines

define inner product for functions

$u(x)$ $v(x)$ on $[-\ell, \ell]$

$$\boxed{(u,v) = \int_{-\ell}^{\ell} u(x)\,v(x)\,dx}$$ function inner prod.

$$\left(\sin\left(\frac{N\pi x}{\ell}\right), \sin\left(\frac{m\pi x}{\ell}\right)\right) = \ell\,\delta_{Nm}$$

$$\left(\cos\left(\frac{N\pi x}{\ell}\right), \cos\left(\frac{m\pi x}{\ell}\right)\right) = \ell\,\delta_{Nm}$$

$$\left(\cos\left(\frac{N\pi x}{\ell}\right), \sin\left(\frac{m\pi x}{\ell}\right)\right) = 0$$

future HW

This means that the basis funct. in Fourier series are orthogonal

Recall

$$f(x) = \frac{a_0}{2} + \sum a_n \cos\left(\frac{n\pi x}{\ell}\right) + b_n \sin\left(\frac{n\pi x}{\ell}\right)$$

or $$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/\ell}$$

$$\left.\begin{aligned} c_k &= \tfrac{1}{2}(a_k - i b_k) \\ c_{-k} &= \tfrac{1}{2}(a_k + i b_k) \end{aligned}\right\}$$ Future HW

$$f(x) = \frac{a_0}{2} + \sum a_n \cos\left(\frac{n\pi x}{\ell}\right) + b_n \sin\left(\frac{n\pi x}{\ell}\right)$$

$$\frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\left(\frac{n\pi x}{\ell}\right)dx = a_n$$

$$\frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)dx = b_n$$

$$\frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\,dx = \frac{1}{\ell}\int_{-\ell}^{\ell}\frac{a_0}{2}\,dx = a_0$$

Eg. $f(x) = |x|$

$$a_0 = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\,dx = \frac{2}{\ell}\int_0^{\ell} x\,dx$$

$$= \frac{2}{\ell}\left.\frac{x^2}{2}\right|_0^{\ell}$$

$$= \ell$$

$a_0 = \ell$

$$a_1 = \frac{1}{\ell}\int_{-\ell}^{\ell} |x| \cos\left(\frac{\pi x}{\ell}\right) dx$$

$$= 2\frac{1}{\ell} \int_0^{\ell} x \cos\left(\frac{n\pi x}{\ell}\right) dx$$

$$\frac{\pi x}{\ell} = z \qquad x = \frac{\ell}{\pi} z$$

$$a_1 = \frac{2}{\ell}\left(\frac{\ell}{\pi}\right)^2 \underbrace{\int_0^{\pi} z \cos z \, dz}_{= -2}$$

$$a_1 = -\frac{4\ell}{\pi^2}$$

$$f(x) = \frac{\ell}{2} - \frac{4\ell}{\pi^2} \cos\left(\frac{\pi x}{\ell}\right) \;\; \ldots\ldots$$

Digression functions as vect.

first discrete infinite dimensional vectors [sequences]

$(a_0, a_1, a_2, \ldots a_n, \ldots b_1, b_2 \ldots b_N \ldots)$

coef. of fourier series, we can add such vectors element by element.

$f(x)$ on $[-\ell, \ell]$ spec. by a continuously infinite # of points.

we can add such vectors point by point.

$$f(x) + g(x)$$