8

Quasi-phase-matching
NL conversion with focusing beams

Reading for this section:

2.4, 2.10

Scaling the NL equations

- Using scaled variables can simplify the NL equations and highlight the characteristic parameters
- SHG example $\omega_2 = 2\omega_1 \quad \Delta k = 2k_1 - k_2$ 2.7.10, 2.7.11

$$\frac{dA_1}{dz} = \frac{2\,i\,d_{eff}\omega_1^2}{k_1 c^2} A_2 A_1^* e^{-i\,\Delta k\,z} \qquad \frac{dA_2}{dz} = \frac{i\,d_{eff}\omega_2^2}{k_2 c^2} A_1^2 e^{+i\,\Delta k\,z}$$

- Scale the fields to the total intensity

$$I_j = 2n_j\varepsilon_0 c\left|A_j\right|^2 \quad a_j = u_j e^{i\phi_j}$$

$$\left|a_j\right|^2 = u_j^2 = \frac{I_j}{I} = \frac{2n_j\varepsilon_0 c}{I}\left|A_j\right|^2 \qquad A_j = u_j\sqrt{\frac{I}{2n_j\varepsilon_0 c}}\,e^{i\phi_j}$$

Writing dimensionless equations

- Rewrite equations with new variables

$$\frac{dA_1}{dz} = \frac{2\,i\,d_{eff}\omega_1^2}{k_1 c^2} A_2 A_1^* e^{-i\,\Delta k\,z} \quad A_j = \sqrt{\frac{I}{2n_j\varepsilon_0 c}}\,u_j e^{i\phi_j} = \sqrt{\frac{I}{2n_j\varepsilon_0 c}}\,a_j$$

$$\frac{d}{dz}\left(u_1\sqrt{\frac{I}{2n_1\varepsilon_0 c}}\,e^{i\phi_1}\right) = \frac{2\,i\,d_{eff}\omega_1}{n_1 c} u_2\sqrt{\frac{I}{2n_2\varepsilon_0 c}}\,e^{i\phi_2} u_1\sqrt{\frac{I}{2n_1\varepsilon_0 c}}\,e^{-i\phi_1} e^{-i\,\Delta k\,z}$$

$$\boxed{\frac{da_1}{dz} = i\frac{2d_{eff}\omega_1}{n_1 c}\sqrt{\frac{I}{2n_2\varepsilon_0 c}}\,a_2\,a_1^* e^{-i\,\Delta k\,z}}$$

$$\frac{dA_2}{dz} = \frac{i\,d_{eff}\omega_2^2}{k_2 c^2} A_1^2 e^{+i\,\Delta k\,z} \qquad \sqrt{\frac{I}{2n_2\varepsilon_0 c}}\frac{da_2}{dz} = i\frac{d_{eff}\,2\omega_1}{n_2 c}\frac{I}{2n_1\varepsilon_0 c}a_1^2 e^{+i\,\Delta k\,z}$$

$$\boxed{\frac{da_2}{dz} = i\frac{2d_{eff}\omega_1}{n_1 c}\sqrt{\frac{I}{2n_2\varepsilon_0 c}}\,a_1^2 e^{+i\,\Delta k\,z}} \qquad l = \frac{c}{2\omega_1 d_{eff}}\sqrt{\frac{2n_1^2 n_2\varepsilon_0 c}{I}}$$

Final form of scaled equations for SHG

$$\frac{da_1}{dz} = i\frac{1}{l}a_2\,a_1^*e^{-i\Delta k z} \qquad \frac{da_2}{dz} = i\frac{1}{l}a_1^2e^{+i\Delta k z} \qquad |a_1|^2 + |a_2|^2 = u_1^2 + u_2^2 = 1$$

- Define dimensionless distance variable

$$\xi = z/l$$

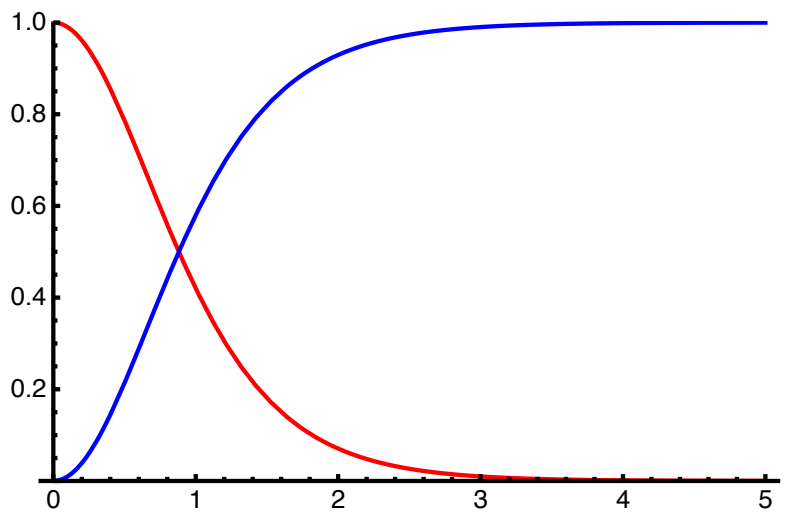
$$\Delta k\,z = \Delta k\,l\,\xi \equiv \Delta s\,\xi$$

$$\boxed{\frac{da_1}{d\xi} = i\,a_2\,a_1^*e^{-i\Delta s\xi}} \qquad \boxed{\frac{da_2}{d\xi} = i\,a_1^2e^{+i\Delta s\xi}}$$

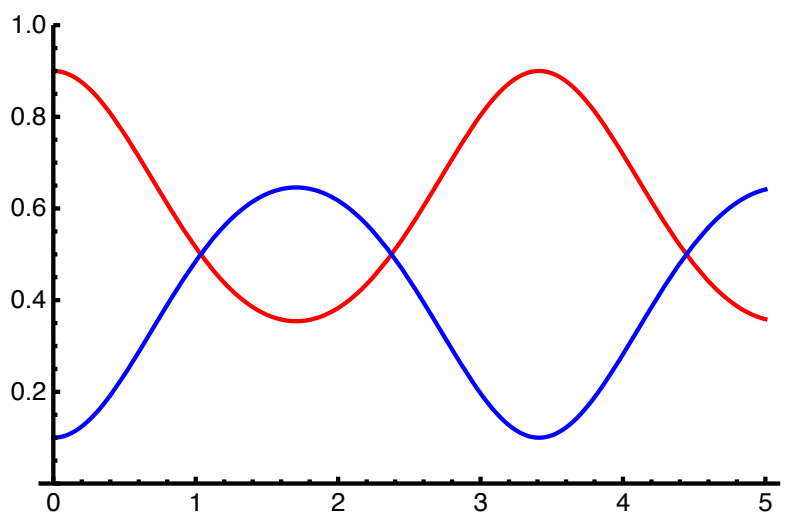
- *l* is the characteristic distance for energy exchange (saturation and back-conversion)
- If there is no SH present at start, full conversion

Saturated SHG conversion

- No seed SH



- With seed



Quasi-phase matching

- Some materials don't support birefringent phase matching
  - $LiNbO_3$ has a strong NL coefficient but in the same vector direction as the input polarization
  - Isotropic materials, e.g. gas or liquid
- Structuring the medium can allow build-up of NL signal without complete phase matching

$$\frac{dA_3}{dz} = \frac{2\,i\,d_{eff}\omega_3^2}{k_3 c^2} A_1 A_2 e^{+i\,\Delta k\,z} \qquad d_{eff}(z) = d_0 \cos K\,z$$

$$\frac{dA_3}{dz} = i\,\frac{d_0\omega_3^2}{k_3 c^2} A_1 A_2 \left(e^{+i\,K\,z} + e^{-i\,K\,z}\right) e^{+i\,\Delta k\,z}$$

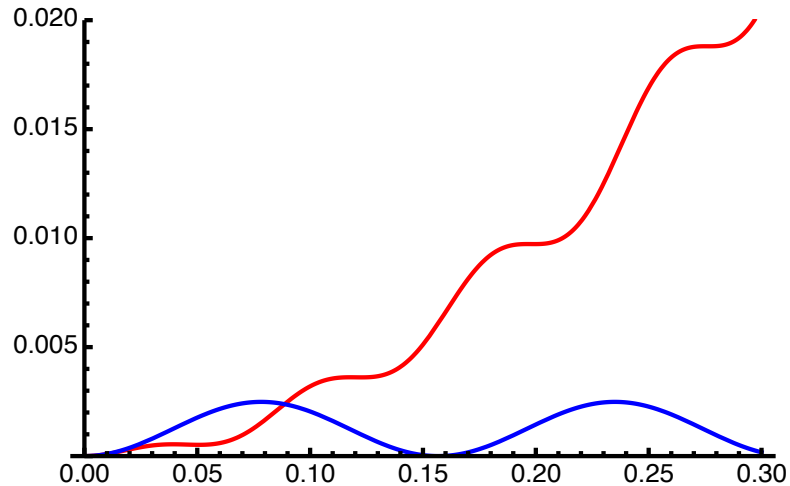
$$= i\,\frac{d_0\omega_3^2}{k_3 c^2} A_1 A_2 \left(e^{+i(K+\Delta k)z} + e^{-i(K-\Delta k)z}\right)$$

if $K \pm \Delta k = 0$,

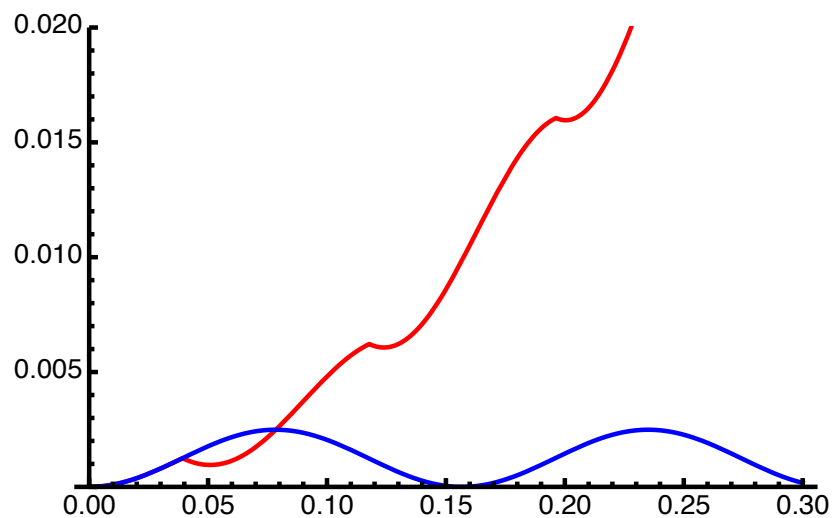
signal can build up

QPM build-up

- QPM cos( ) modulation:



- QPM, Sign(cos()) modulation (layered):
  - Calculate fourier series of modulation function, pick out component that can cancel phase mismatch



3D propagation

$$\nabla^2 \mathbf{E_j} - \frac{n_j^2}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E_j} = \frac{\partial^2}{\partial z^2}\mathbf{E_j} + \nabla_\perp^2 \mathbf{E_j} - \frac{n_j^2}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E_j} = \frac{1}{\varepsilon_0 c^2}\frac{\partial^2}{\partial t^2}\mathbf{P_j^{NL}}$$

- Notes:
  - RHS is source term
  - All linear propagation effects are included in LHS: diffraction, interference, focusing…

$$\nabla_\perp^2 = \partial_x^2 + \partial_y^2$$

$$\boxed{\nabla_\perp^2 = \frac{1}{r}\partial_r\left(r\,\partial_r\right) + \frac{1}{r^2}\partial_\phi^2}$$

- So far, we've assumed plane waves where transverse derivatives are zero.
- Counter examples:
  - Gaussian beams (including high-order)
  - Waveguides
  - Arbitrary propagation
- Often determine solutions to linear equation (e.g. Gaussian beams, waveguide modes), then express fields in terms of those solutions.

Paraxial, slowly-varying

- Assume waves are forward-propagating:

$$\mathbf{E}_j(\mathbf{r},t) = \mathbf{A}_j(\mathbf{r}) e^{i(k_j z - \omega_j t)} + \text{c.c.}$$

$$\mathbf{P}_j(\mathbf{r},t) = \mathbf{p}_j(\mathbf{r}) e^{i(k'_j z - \omega_j t)} + \text{c.c.}$$

$$\frac{\partial^2}{\partial z^2}\mathbf{A_j} + 2ik_j \frac{\partial}{\partial z}\mathbf{A_j} - k_j^2 \mathbf{A_j} + \nabla_\perp^2 \mathbf{A_j} + \frac{n_j^2 \omega_j^2}{c^2}\mathbf{A_j} = -\frac{\omega_j^2}{\varepsilon_0 c^2}\mathbf{p_j} e^{i\Delta k z}$$

  - Fast oscillating carrier terms cancel (blue)
- Slowly-varying envelope: compare red terms
  - Drop 2nd order deriv if $\frac{2\pi}{\lambda_j}\frac{1}{L}A_j \gg \frac{1}{L^2}A_j$
  - Ignoring any counterpropagating waves

Gaussian beam solutions to wave equation

- Without any source term, paraxial equation is

$$2ik_j \frac{\partial}{\partial z}\mathbf{A_j} + \nabla_\perp^2 \mathbf{A_j} = 0$$

- Gaussian beam solutions can be written as:

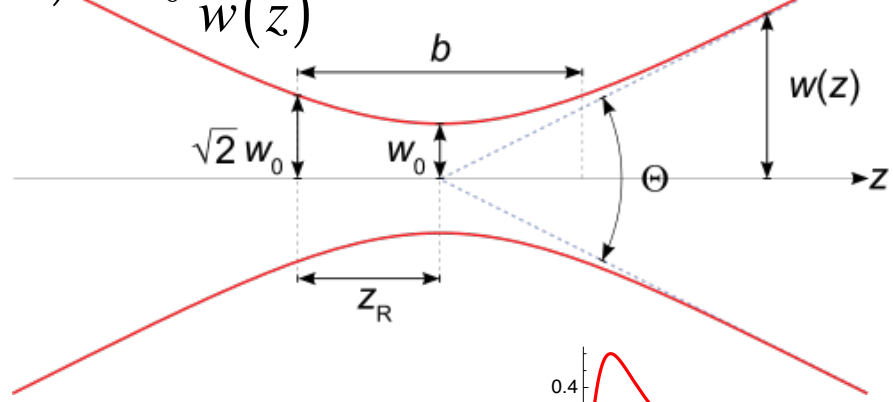
$$A(r,z) = A_0 \frac{1}{1+i\xi} e^{-\frac{r^2}{w_0^2(1+i\xi)}} \qquad \xi = \frac{z}{z_R} \qquad z_R = n\frac{\pi w_0^2}{\lambda} = \frac{k\,w_0^2}{2}$$

Rayleigh range

Gaussian beam propagation equations

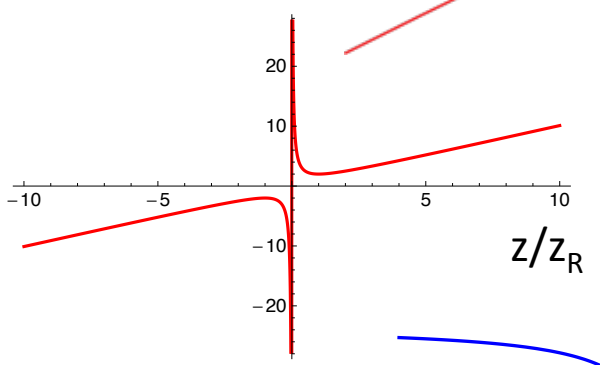
- Standard form

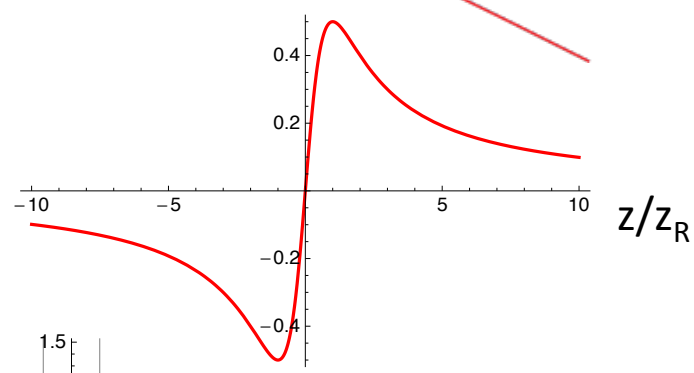
$$E(r,z,t) = A_0 \frac{w_0}{w(z)} e^{i(kz-\omega t)} e^{-\frac{r^2}{w^2(z)}} e^{i\frac{kr^2}{2R(z)}} e^{-i\eta(z)}$$



$$w(z) = w_0 \sqrt{1 + \frac{z^2}{z_R^2}}$$

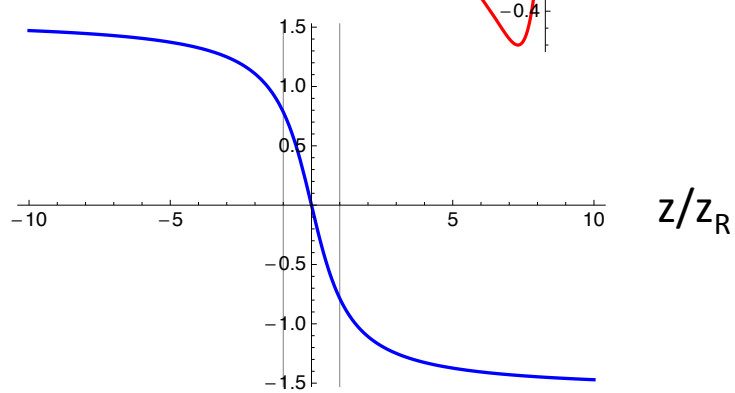
$$R(z) = z\left(1 + \frac{z_R^2}{z^2}\right)$$





Gouy phase

$$\eta(z) = \arctan\left(\frac{z}{z_R}\right)$$



Gaussian beam propagation equations

- Complex q form for ABCD (Siegman form, exp[+i w t])

$$\frac{1}{q(z)} = \frac{1}{R(z)} - i\frac{\lambda}{\pi w^2(z)}$$

$$\frac{1}{q(z)} = \frac{1}{z + i z_R} = \frac{z}{z^2 + z_R^2} - i\frac{z_R}{z^2 + z_R^2}$$

$$= \frac{1}{R(z)} - i\frac{w_0^2}{z_R w^2(z)}$$

$$= \frac{1}{R(z)} - i\frac{\lambda}{\pi w^2(z)} = \frac{1}{R(z)} - i\frac{1}{Z(z)}$$

$$\frac{1}{R(z)} = \frac{1}{z\left(1 + \frac{z_R^2}{z^2}\right)} = \frac{z}{z^2 + z_R^2}$$

$$\frac{1}{w^2(z)} = \frac{1}{w_0^2\left(1 + \frac{z^2}{z_R^2}\right)} = \frac{z_R^2}{w_0^2\left(z^2 + z_R^2\right)}$$

$$u(r,z) = \frac{1}{q(z)} e^{-ik\frac{r^2}{2q(z)}}$$

Complex q vs standard form

$$u(r,z) = \frac{1}{q(z)} e^{-ik\frac{r^2}{2q(z)}} \qquad \text{with} \qquad \frac{1}{q(z)} = \frac{1}{R(z)} - i\frac{\lambda}{\pi w^2(z)}$$

Expand exponential:

$$\exp\left[-ik\frac{r^2}{2q(z)}\right] = \exp\left[-ik\frac{r^2}{2}\left(\frac{1}{R(z)} - i\frac{\lambda}{\pi w^2(z)}\right)\right]$$

$$= \exp\left[-ik\frac{r^2}{2}\frac{1}{R(z)} - i\frac{2\pi}{\lambda}\frac{r^2}{2}\left(-i\frac{\lambda}{\pi w^2(z)}\right)\right] = e^{-ik\frac{r^2}{2R(z)}} e^{-\frac{r^2}{w^2(z)}}$$

$$a + ib = \sqrt{a^2+b^2}\, e^{i\arctan(b/a)}$$

Exand leading inverse q:

$$\frac{1}{q(z)} = -i\left(\frac{iz}{z^2+z_R^2} + \frac{z_R}{z^2+z_R^2}\right) = -i\left(\frac{\sqrt{z^2+z_R^2}}{z^2+z_R^2}\right) e^{i\arctan(z/z_R)}$$

$$= -i\left(\frac{1}{z_R\sqrt{1+z^2/z_R^2}}\right) e^{i\arctan(z/z_R)} = \frac{w_0}{i z_R w(z)} e^{i\eta(z)}$$

Difference between Siegman's complex q and standard form

$$u(r,z) = \frac{1}{q(z)} e^{-ik\frac{r^2}{2q(z)}} = \frac{1}{i z_R} \frac{w_0}{w(z)} e^{i\eta(z)} e^{-ik\frac{r^2}{2R(z)}} e^{-\frac{r^2}{w^2(z)}}$$

$$E(r,z,t) = A_0 \frac{w_0}{w(z)} e^{i(kz-\omega t)} e^{-\frac{r^2}{w^2(z)}} e^{i\frac{kr^2}{2R(z)}} e^{-i\eta(z)}$$

- Siegman's form for the complex q is used almost everywhere for the ABCD calculations.
- He uses the exp[+ I w t] convention, which accounts for the sign difference in the complex exponentials.
- With exp[-I w t] convention, define q as:

$$\frac{1}{q(z)} = \frac{1}{R(z)} + i\frac{\lambda}{\pi w^2(z)} = \frac{1}{z - i z_R}$$

Compare Boyd's form to standard:

- Boyd's complex form is consistent with standard Gaussian beam form

$$A(r,z) = A_0 \frac{1}{1+i\xi} e^{-\frac{r^2}{w_0^2(1+i\xi)}} = A_0 \frac{1}{1+i\,z/z_R} e^{-\frac{r^2}{w_0^2(1+i\,z/z_R)}}$$

$$\frac{1}{1+i\xi} = \frac{1}{1+i\,z/z_R} = \frac{z_R}{z_R+i\,z} = \frac{-i\,z_R}{z-i\,z_R} = \frac{-i\,z_R}{q(z)}$$

$$A(r,z) = A_0(-i\,z_R)\frac{1}{q(z)} e^{+\frac{i\,z_R r^2}{w_0^2 q(z)}} = -i\,z_R A_0 \frac{1}{q(z)} e^{+\frac{i\,k\,r^2}{2\,q(z)}}$$

Harmonic generation with focused Gaussian beams

- q^{th} harmonic, no depletion

$$\omega_q = q\omega_1 \qquad 2ik_q \frac{\partial}{\partial z} A_q + \nabla_\perp^2 A_q = -\frac{\omega_q^2}{\varepsilon_0 c^2} \chi^{(q)} A_1^q e^{i\Delta k z}$$

- *Assume* harmonic propagates as a TEM_{00} beam

$$A_q(r,z) = A_{q0}(z) \frac{1}{1+i\xi_q} e^{-\frac{r^2}{w_{q0}^2(1+i\xi_q)}}$$

$$A_{q0}(z) \cancel{\left(2ik_q \frac{\partial}{\partial z} + \nabla_\perp^2\right) \frac{1}{1+i\xi_q} e^{-\frac{r^2}{w_{q0}^2(1+i\xi_q)}}} + \frac{1}{1+i\xi_q} e^{-\frac{r^2}{w_{q0}^2(1+i\xi_q)}} \frac{\partial}{\partial z} A_{q0}(z)$$

$$\frac{1}{1+i\xi_q} e^{-\frac{r^2}{w_{q0}^2(1+i\xi_q)}} \frac{\partial}{\partial z} A_{q0}(z) = -\frac{\omega_q^2}{2ik_q \varepsilon_0 c^2} \chi^{(q)} \left(\frac{1}{1+i\xi_1}\right)^q e^{-\frac{qr^2}{w_0^2(1+i\xi_1)}} A_1^q e^{i\Delta k z}$$

$$\frac{\partial}{\partial z} A_{q0}(z) = -\frac{\omega_q^2}{2ik_q \varepsilon_0 c^2} \chi^{(q)} \frac{1+i\xi_q}{(1+i\xi_1)^q} e^{-\frac{qr^2}{w_0^2(1+i\xi_1)} + \frac{r^2}{w_{q0}^2(1+i\xi_q)}} A_1^q e^{i\Delta k z}$$

Phase matching integral, non-depleted limit

- With both Gaussian beams, matched Rayleigh ranges

$$\frac{\partial}{\partial z}A_{q0}(z) = -\frac{\omega_q^2}{2ik_q\varepsilon_0 c^2}\chi^{(q)}\frac{1+i\xi_q}{(1+i\xi_1)^q}e^{-\frac{qr^2}{w_0^2(1+i\xi_1)}+\frac{r^2}{w_{q0}^2(1+i\xi_q)}}A_1^q e^{i\Delta k z}$$

If $w_{q0}^2 = w_0^2/q$ And letting $n_q \approx n_1$ $\rightarrow \frac{\pi w_{q0}^2}{\lambda_q} = \frac{\pi w_0^2/q}{\lambda_1/q}$

$$z_R(\omega_q) = z_R(\omega_1) \qquad \xi_q = \xi_1$$

$$\rightarrow \frac{\partial}{\partial z}A_{q0}(z) = i\frac{\omega_q}{2n_q\varepsilon_0 c}\chi^{(q)}\frac{1}{(1+i\xi)^{q-1}}A_1^q e^{i\Delta k z}$$

Integrate directly to get

$$\rightarrow A_{q0}(z) = i\frac{\omega_q}{2n_q\varepsilon_0 c}\chi^{(q)}A_1^q J_q(z)$$

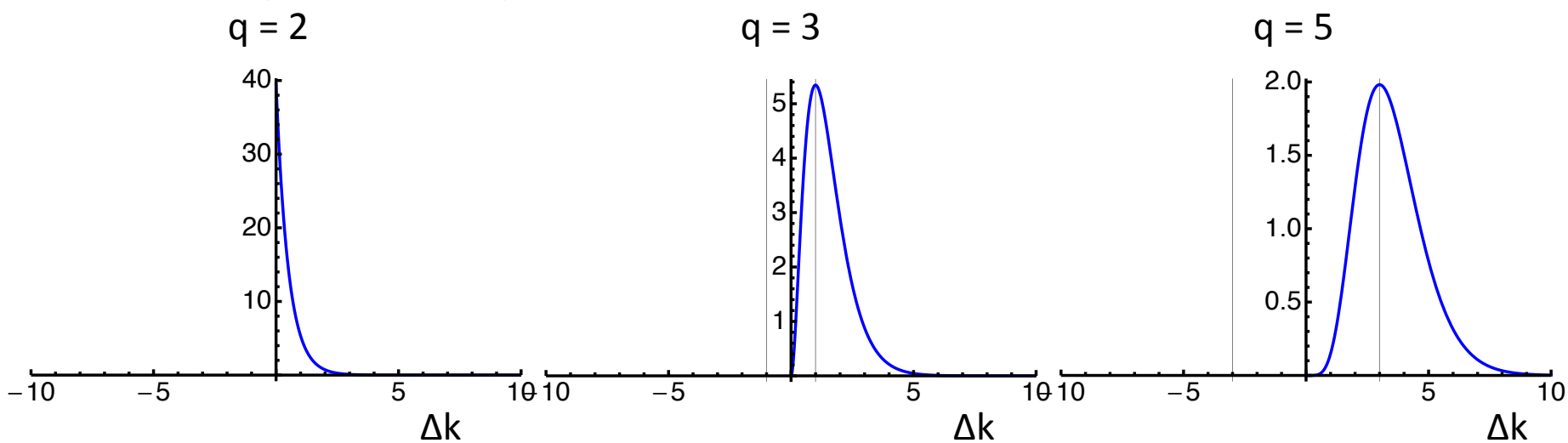
$$J_q(z) = \int_{z_1}^{z_2}\frac{1}{(1+iz'/z_R)^{q-1}}e^{i\Delta k z'}dz'$$

Tight focusing limit

- Here we integrate over all z:

$$J_q\left(\Delta k, z_R\right) = \begin{cases} 0 & \Delta k < 0 \\ z_R \dfrac{2\pi}{(q-2)!}(\Delta k\, z_R)^{q-2} e^{-\Delta k\, z_R} & \Delta k \geq 0 \end{cases}$$

- For q>2, zero yield unless Δk>0



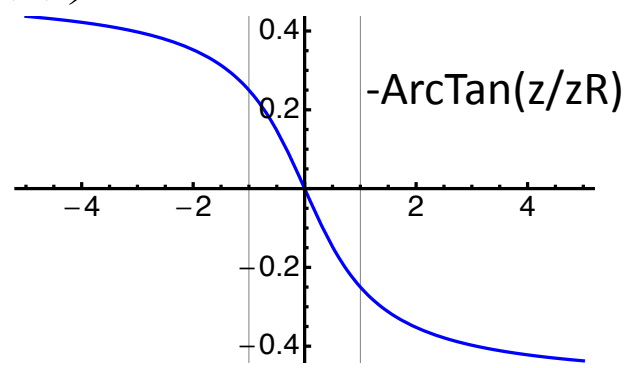
Small thickness limit

$$J_q(z) = \int_{-L/2}^{L/2} \frac{1}{\left(1 + i z' / z_R\right)^{q-1}} e^{i \Delta k z'} dz'$$

The fraction in the integrand is connected to the Gouy phase:

$$\frac{1}{1+i\xi} = \frac{1}{1+i z / z_R} = \frac{w_0}{w(z)} e^{-i\eta(z)} \qquad \eta(z) = \arctan\left(\frac{z}{z_R}\right)$$

$$\frac{1}{\left(1+i z' / z_R\right)^{q-1}} = \frac{1}{\left(1+i\xi\right)^{q-1}} = \left(\frac{w_0}{w(z)}\right)^{q-1} e^{-i(q-1)\eta(z)} \approx \left(\frac{w_0}{w(z)}\right)^{q-1} e^{-i(q-1)z/z_R}$$

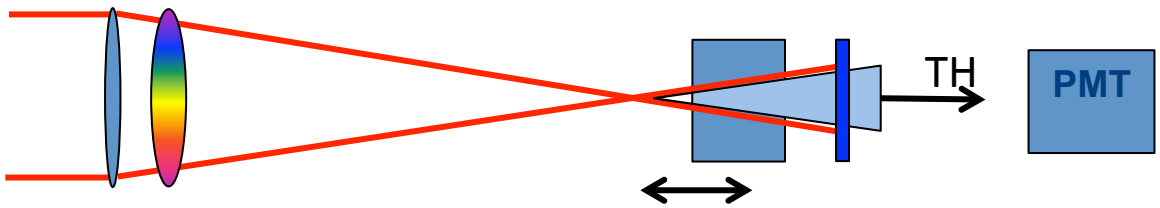


$$J_q(z) = \int_{-L/2}^{L/2} \left(\frac{w_0}{w(z)}\right)^{q-1} e^{i\left(\Delta k - (q-1)/z_R\right) z'} dz'$$

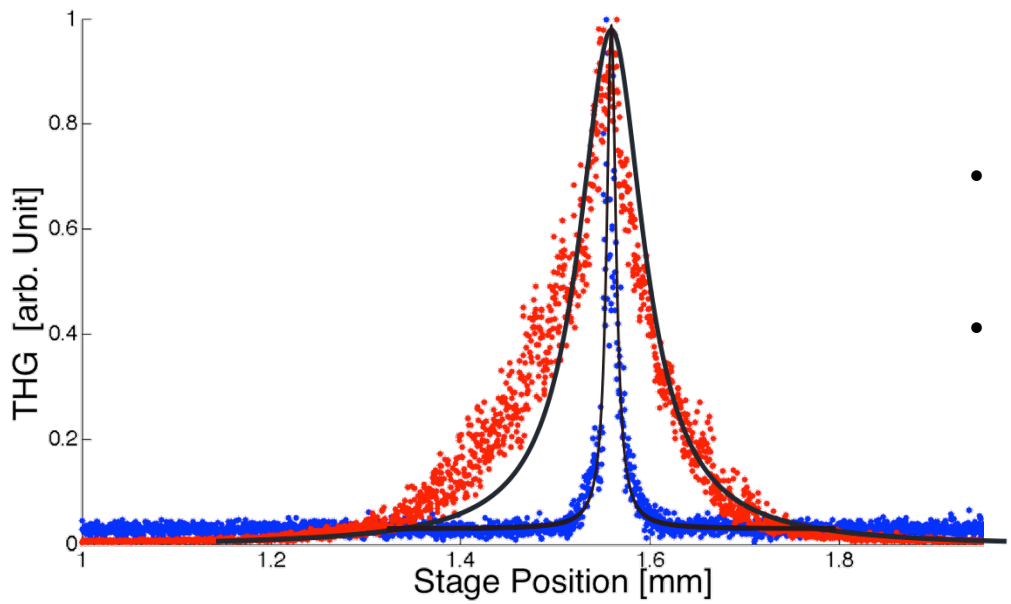
This shows where the optimum phase mismatch should be.
This limit is related to HG in waveguides, since the WG phase scales like z/zR

Measuring localization with THG

- Z-scan of fused silica interface leads to observed THG through partial phase matching



- No THG from bulk
- THG emerges with spatial chirp



- Axial FWHM = confocal parameter of fundamental in air
- SSTF reduces FWHM consistent with beam aspect ratio

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