

Day 29: Motional EMF + Faraday's Law I

"Mr. Faraday, of what use are your discoveries?"

"Madam, of what use is a baby?" - Michael Faraday

We're now moving into full-blown electrodynamics, where there will almost always be some time-dependence somewhere.

We need to start with a little vocabulary:

EMF: electromotive force; a measure of how much work a system is doing on a unit charge. Has units of volts, or J/C (energy/charge)

Physically very much like voltage, though "voltage" or "electric potential" tend to be used in situations where a scalar potential can be defined.

Since work is $\int \vec{F} \cdot d\vec{\ell}$ and the Lorentz force law is $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$, EMF will be of the form $\int \vec{F}/q \cdot d\vec{\ell} = \int (\vec{E} + \vec{v} \times \vec{B}) \cdot d\vec{\ell}$

Generally we focus on either electric or magnetic forces.

We've seen Faraday's Law written two different ways in Phys 200:

$$\oint (\vec{E} + \vec{v} \times \vec{B}) \cdot d\vec{\ell} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{A}$$

$$\text{EMF} = -\frac{d\Phi}{dt}$$

And we're going to add a third:

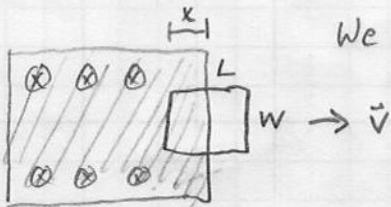
$$\vec{\nabla} \times \vec{E} = -\frac{d\vec{B}}{dt}$$

(Clicker question)

Let's work through a few different situations and see what kind of insight we can obtain.

EMF from a static B-field

Let's suppose we have a B-field that occupies some region of space. It's uniform, with magnitude B , into the page. B is zero everywhere else. (This field is actually unphysical - do you see why?)



We also have a wire loop with dimensions $w \times L$ partly in the field. We're pulling it out with some speed v . x is the length of the loop still in the field.

The charge carriers in the wire (let's call them positive for convenience) feel an EMF due to the Lorentz force:

$$\text{EMF} = \oint (\vec{v} \times \vec{B}) \cdot d\vec{l}$$

$$\Rightarrow \text{EMF} = vBw$$

Now, the right hand rule shows us that only charges in the leftmost wire segment are acted on (only there is $\vec{v} \times \vec{B} \parallel$ to $d\vec{l}$)

Let's look at $\frac{d\Phi}{dt}$ also. $\Phi = \int \vec{B} \cdot d\vec{A}$, or in this case, Bwx

$$\text{Thus } \frac{d\Phi}{dt} = \frac{d}{dt}(Bwx) = Bwv$$

So we find that

$$\text{EMF} = \oint (\vec{v} \times \vec{B}) \cdot d\vec{l} = -\frac{d\Phi}{dt} \quad (1)$$

(What's that sign about?)

You'll notice there is no \vec{E} anywhere in the problem.

This turns out to be a general result for any moving loop in a B-field. You can prove it using $\vec{\nabla} \cdot \vec{B} = 0$ and $\vec{F} = q\vec{v} \times \vec{B}$ (see section 10.1.2 in Plick & Stump)

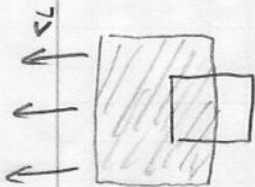
To emphasize for future use:

$$\vec{\nabla} \cdot \vec{B} = 0 + \vec{F} = q\vec{v} \times \vec{B}$$
$$\Rightarrow \text{EMF} = -\frac{d\Phi}{dt}$$

With no E , and being derivable from existing principles, this is not Faraday's Law (not when the EMF comes from a strictly magnetic force. But it is quite closely related.

Electromagnetic Induction + Faraday's (actual) Law

Now, what of the reciprocal situation where the wire loop is stationary and the region occupied by the field is moving, thus providing a time varying \vec{B} ?



We cannot derive what happens from what we currently know. We need a brand new, empirically obtained law: Faraday's Law in differential form:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

This law tells us that E-fields have another source besides charges: time-varying magnetic fields

This law will let us figure out what happens in any situation where \vec{B} is changing, thus inducing a curly EMF such that $\oint \vec{E} \cdot d\vec{l} \neq 0$

And so we come full circle.

We have to proceed carefully, though. Many people (and books) screw up what follows as we try to get Faraday's Law into integral form. First, integrate both sides over some open surface:

$$\int (\nabla \times \vec{E}) \cdot d\vec{A} = - \int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{A} \quad \text{Stokes' thm on left}$$

$$\oint \vec{E} \cdot d\vec{l} = - \int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{A}$$

Right hand side is a spatial integral only, so we can pull out $\frac{\partial}{\partial t}$ and make it a total derivative

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{A}$$

That seemed simple enough. And it's true that $\nabla \times \vec{E} = -\partial \vec{B} / \partial t$ and $\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{A}$ are equivalent if and only if \vec{B} is the only thing with a time dependence.

What happens if we go back to the first situation, where \vec{B} isn't changing but the loop is moving? Start with:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Integrate both sides and apply Stokes on left:

$$(1a) \quad \oint \vec{E} \cdot d\vec{\ell} = - \int \frac{d\vec{B}}{dt} \cdot d\vec{A} \quad \text{Now we don't pull out } \frac{d}{dt}$$

Why not?

Ok, so what we have here is that $\vec{\nabla} \times \vec{E} = - \frac{d\vec{B}}{dt}$ and the above are (1a) equivalent with no restrictions whatsoever. But we can't necessarily get that d/dt out if the area is changing, because the bounds of the integral are time dependent!

$$(2) \quad \oint \vec{E} \cdot d\vec{\ell} = - \int_{S(t)} \frac{d\vec{B}}{dt} \cdot d\vec{A} \quad \text{Note that this does give a consistent result in the case of the moving loop, since } \frac{d\vec{B}}{dt} = 0 \text{ and } \vec{E} = 0 \text{ there.}$$

But can we do better? What can we do with that $\int \frac{d\vec{B}}{dt} \cdot d\vec{A}$ term? Let's look at

$$- \frac{d}{dt} \int_{S(t)} \vec{B} \cdot d\vec{A} \quad \text{and see if we can relate it.}$$

You take the derivative of functions whose bounds have variables using the Leibniz rules. You've seen one in 1-D:

$$\frac{d}{dt} \int_{a(t)}^{b(t)} f(x,t) dx = \frac{db}{dt} f(b,t) - \frac{da}{dt} f(a,t) + \int_a^b \frac{d}{dt} f(x,t) dx$$

It's got a vaguely product/chain rule flavor to it, where to apply the derivative completely we need to hit the b-endpoint, the a-endpoint, and the integrand. This generalizes to vector fields instead of scalar functions, but not cleanly. Letting \vec{B} be our vector field, the 3-D Leibniz rule gives:

$$\frac{d}{dt} \int \vec{B}(\vec{x},t) \cdot d\vec{A} = \int \left(\frac{d\vec{B}}{dt} + (\vec{\nabla} \cdot \vec{B}) \vec{v} \right) \cdot d\vec{A} - \oint (\vec{v} \times \vec{B}) \cdot d\vec{\ell}$$

Where \vec{v} comes from the time derivative of the spatial coordinates defining the area (so, for example, the velocity of the wire loop).

$\vec{\nabla} \cdot \vec{B} = 0$, so we can find that

$$- \int \frac{d\vec{B}}{dt} \cdot d\vec{A} = - \oint (\vec{v} \times \vec{B}) \cdot d\vec{\ell} - \frac{d}{dt} \int \vec{B}(\vec{x},t) \cdot d\vec{A}$$

Substituting this into (2) yields

$$\oint \vec{E} \cdot d\vec{l} = -\oint (\vec{v} \times \vec{B}) \cdot d\vec{l} - \frac{d}{dt} \int \vec{B} \cdot d\vec{A}$$

$$\Rightarrow \oint (\vec{E} + \vec{v} \times \vec{B}) \cdot d\vec{l} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{A}$$

Which is true generally. It is basically a version of

$$EMF = -\frac{d\Phi}{dt} \quad \text{that shows explicitly how } \vec{E} + \vec{B}$$

can both contribute to the EMF.

So that leaves us with five forms of Faraday's Law:

$$\nabla \times \vec{E} = -\frac{d\vec{B}}{dt}$$

Faraday's law proper, in differential form. Relates time varying B-fields and curly E-fields at single points in space.

$$\oint \vec{E} \cdot d\vec{l} = -\int \frac{d\vec{B}}{dt} \cdot d\vec{A}$$

Integral form of the above. Always true. Relates $E+B$ over entire regions.

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{A}$$

True only if the loop in question is entirely static. If misapplied generally, forces you to conclude that there's a curly E-field present anytime there's a time varying flux for any reason. This can lead to some serious, substantive errors - it's not mere pedantry.

$$\oint (\vec{E} + \vec{v} \times \vec{B}) \cdot d\vec{l} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{A}$$

Corrected form of the above. Adding a motional EMF term fixes it.

$$EMF = -\frac{d\Phi}{dt}$$

A field-agnostic version of "Faraday's Law". Says time varying magnetic fluxes are associated with voltages. Doesn't actually say time varying B-fields have anything to do with curly E-fields. So is it really Faraday's Law? Seems like "Faraday's Law" has

evolved into kind of an umbrella term. And that's okay, if you understand the bits & pieces.