Electrostatics: Coulomb's & Gauss's laws

The general set of equations governing electromagnetism is as follows:

$$\nabla \cdot \vec{E} = \rho/\varepsilon_0$$

$$\nabla \cdot \vec{B} = 0 \qquad\qquad \vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

But we don't generally try to tackle that all at once. We'll get started by studying what happens when all the charges in the neighborhood are fixed in place - what we call <u>electrostatics</u>. Then all the time varying terms go away, as do the currents, and in fact <u>all</u> 'sources' of B-fields, leaving us with just:

$$\nabla \cdot \vec{E} = \rho/\varepsilon_0 \qquad\qquad \vec{F} = q\vec{E}$$

$$\nabla \times \vec{E} = 0$$

And as an added bonus, you can even derive $\nabla \times \vec{E} = 0$ from Gauss's law, leaving us with only two unique equations. We might look at that derivation later on if we have time.

Now, you may not be accustomed to seeing Gauss's law in differential form, so let's convert it to integral form. I'll integrate over some volume:

$$\int (\nabla \cdot \vec{E})\, d^3x = \int \rho/\varepsilon_0 \, d^3x$$

Note that there are several conventions for writing a differential volume element. dV is not preferred since we sometimes use it for a differential voltage. You'll see d^3x, dv (lower case), and even $d\tau$

On the right, integrating a volume charge density over a volume gives a charge:

$$\int (\nabla \cdot \vec{E})\, d^3x = \frac{1}{\varepsilon_0}\int \rho \, d^3x = \frac{Q}{\varepsilon_0}$$

On the left, we can apply the divergence theorem:

$$\int (\nabla \cdot \vec{E})\, d^3x = \oint \vec{E} \cdot d\vec{A}$$

And we obtain the familiar intro version of Gauss's law:

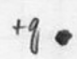
$$\boxed{\oint \vec{E} \cdot d\vec{A} = Q_{enc}/\varepsilon_0}$$

$$\boxed{\nabla \cdot \vec{E} = \rho/\varepsilon_0}$$

You can also take the integral form and derive from it the differential form. The steps are <u>almost</u> the same but in reverse, <u>except</u> for one tricky part that I won't spoil for you.

These forms are equivalent insofar as you can derive one from the other, but say slightly different things. The differential form is a statement about <u>single points</u> in space - if there's some ρ at a point, there's also a diverging E-field there. The integral form is a statement about whole regions - the flux through some shape is proportional to the amount of charge it encloses.

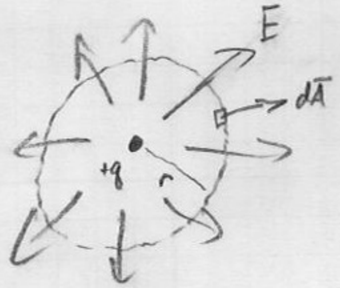
I promised that you can get all of electrostatics from $\nabla \cdot \vec{E} = \rho/\varepsilon_0$ and $\vec{F} = q\vec{E}$, and you can. That includes Coulomb's law, as long as we include one more not-terribly-controversial postulate. Start with a positive point charge of magnitude q at the origin:



If we posit that space is rotationally invariant (changing the orientation of a system doesn't change the physics), we can immediately conclude that a point charge (being spherically symmetric) must produce a strictly radial field. To see this, try a little proof by contradiction:

Suppose q makes some non-radial field at some point. Then suppose we take the whole system and rotate it 180° about the indicated axis

The actual charge is unchanged, but now it makes the opposite field! This is kind of nonsense, and our claim that there could be a non-radial field must have been false.



Ok, so we have a radial field. Let's draw a spherical Gaussian surface (radius r) around it and write down Gauss's law:

$$\oint \bar{E} \cdot d\bar{A} = \frac{Q_{enc}}{\varepsilon_0}$$

We're integrating $\bar{E} \cdot d\bar{A}$ all around that sphere. $d\bar{A}$ is the area vector corresponding to a little patch of that surface, and points radially outward. $\bar{E}$ also points radially outward everywhere, so at any point on the Gaussian surface, $\bar{E} \cdot d\bar{A} = E\,da$

(and $Q_{enc} = q$, naturally)

$$\Rightarrow \oint E\,dA = q/\varepsilon_0$$

Now, this system is rotationally invariant (it has spherical symmetry), so the magnitude E has to be the same everywhere on the domain of integration, and we can pull it out:

$$\Rightarrow E \oint dA = q/\varepsilon_0$$

$\oint dA$ is just A, or $4\pi r^2$ in this case

$$\Rightarrow E \cdot 4\pi r^2 = q/\varepsilon_0$$

$$\Rightarrow E = \frac{q}{4\pi\varepsilon_0 r^2}$$

And we know $\bar{E}$ is radial, so:

$$\boxed{\bar{E}_{point} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r} = \frac{q}{4\pi\varepsilon_0}\frac{\bar{r}}{r^3}}$$

Which is indeed Coulomb's law

Day 2 - Coulomb's + Gauss's Law

Coulomb's Law: Empirically derived: $F = \frac{Kq_1q_2}{r^3}\vec{r}$ or $\frac{Kq_1q_2}{r^2}\hat{r}$

We define electric field via $\vec{E} = \vec{F}/q_0 \Rightarrow \vec{E} = \frac{Kq}{r^2}\hat{r}$ Often still called Coulomb's Law

Superposition applies: $\vec{E}_{NET} = \sum_{i=1}^{N} \frac{Kq_i}{r^2}\hat{r}$

We can generalize to continuous distributions: $d\vec{E} = \frac{Kdq}{r^3}\vec{r}$

Coulomb's Law + Superposition + some math generates <u>all</u> of electrostatics

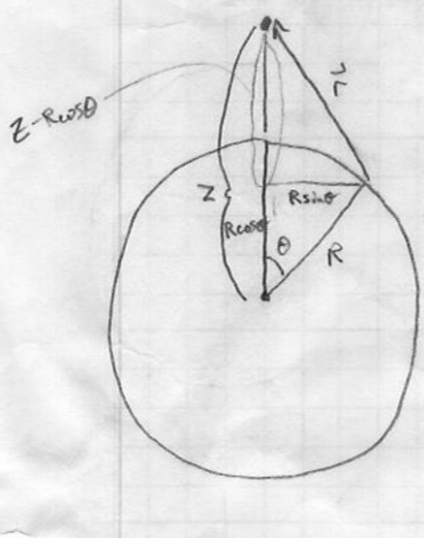
Some warnings (notational): $K = \frac{1}{4\pi\epsilon_0}$ $\quad \vec{r} = \vec{x}_1 - \vec{x}_2$ or $\vec{x} - \vec{x}'$ or even $\vec{\mathscr{r}}$

$dV \leftrightarrow d^3x' \qquad dA \leftrightarrow d^2x'$

(Clicker question)

A nice coulomb's Law derivation. Have seen via Gauss's Law that spherically symmetric charge distributions look like points from outside, and symmetric shells have zero field inside. We'll do this via integration:

First we choose to believe that the spherical symmetry gives us a strictly radial field (more on this later) and as a fun trick, we look at the field at some distance along the z-axis



$\vec{E} = \int \frac{KdQ}{r^3}\vec{r}$ $\qquad dQ = \sigma dA = \sigma R^2 \sin\theta \, d\theta \, d\phi$

$\vec{r}$: only vertical component matters, $\vec{r} \rightarrow (z - R\cos\theta)\hat{k}$

for $r = |\vec{r}|$, both sides matter: $r = \sqrt{(R^2\sin^2\theta) + (z - R\cos\theta)^2}$

$= (R^2\sin^2\theta + z^2 - 2Rz\cos\theta + R^2\cos^2\theta)^{1/2}$

$r^3 = (R^2 + z^2 - 2Rz\cos\theta)^{3/2}$

$$\vec{E} = \int_0^{2\pi}\int_0^{\pi} \frac{K\sigma R^2 \sin\theta \, d\theta \, d\phi \, (z - R\cos\theta)\hat{k}}{(R^2+z^2-2Rz\cos\theta)^{3/2}}$$ do ϕ

$$E = 2\pi K\sigma R^2 \int_0^{\pi} \frac{\sin\theta \, d\theta \, (z - R\cos\theta)}{(R^2+z^2-2Rz\cos\theta)^{3/2}}\hat{k}$$

Sub $u = \cos\theta$, $du = -\sin\theta \, d\theta$

$$\underline{E} = 2\pi k \sigma R^2 \int_1^{-1} \frac{-du(z-Ru)}{(R^2+z^2-2Rzu)^{3/2}}$$

· Tables or machine

$$2\pi k\sigma R^2 \frac{1}{z^2}\left[\frac{z-R}{\sqrt{(z-R)^2}} + 1\right]$$

For $z<R$ get $R-z$

and $\frac{z-R}{R-z} = -1$ $\boxed{E=0}$

For $z>R$ use $z-R$

get $1+1=2$

$$E = \frac{2\pi k\sigma R^2 \cdot 2}{z^2} \qquad \sigma = \frac{Q}{4\pi R^2}$$

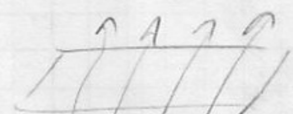
$$\boxed{E = \frac{kQ}{z^2}}$$

Bam!

We've gotten this before with Gauss's Law, and as it turns out

Coulomb's Law $\Rightarrow$ Gauss's Law + vice versa (almost)

Going Coulomb $\rightarrow$ Gauss is hard formally so let's skip derivation for now + recall definition of flux: $\Phi = \int \vec{E}\cdot d\vec{A}$

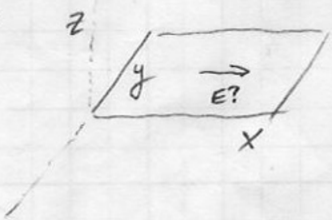


Conceptually, how many field lines go through a surface.

Gauss's Law is a statement about the flux through a closed surface, and is <u>always</u> true. If we have good (planar, cylindrical, spherical) symmetry, we will <u>also</u> be able to solve for E. But we make many remarkably subtle arguments along the way.

Let's do an infinite (homogeneous) sheet, in Cartesian coordinates, charge density σ.

$\oint \vec{E}\cdot d\vec{A} = Q/\epsilon_0$



Suppose $\exists$ ("there exists") a component of $\vec{E}$ in the $\hat{i}$-direction. Now, the charge has a variety of symmetries, including symmetry under reflection + rotation. Twisting the sheet about the z-axis doesn't change it and thus can't change the field, so assuming E_x exists leads to a contradiction. Thus, E_x (and similarly E_y) $= 0$

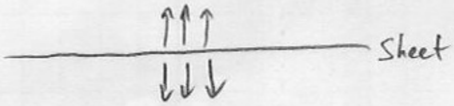
E_z respects all available symmetries and so can exist. Assuming + charge, it'd point up + down. I therefore draw a Gaussian surface like so:

$\oint \vec{E}\cdot d\vec{A} = Q_{enc}/\epsilon_0$ Work left first.

$$\oint \vec{E}\cdot d\vec{A} = \int_{top+bottom} \vec{E}\cdot d\vec{A} = \int_{t+b} E\,dA = E\int_{t+b} dA = E\cdot 2A_{top}$$

Every single step came with a reason. Know them:

Now, $Q_{enc} = \sigma A_{top}$, so $\quad E \cdot 2A_{top} = \frac{\sigma A_{top}}{\varepsilon_0} \Rightarrow \boxed{E_{sheet} = \frac{\sigma}{2\varepsilon_0}}$

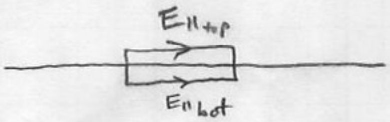


Notice this discontinuity in $E_\perp$!

$$\boxed{E_{\perp top} - E_{\perp bot} = \frac{\sigma}{\varepsilon_0}}$$

Always true
Can always zoom in till the surface is locally flat + draw a tiny box

This is a boundary condition.
It will come back, and it will bring a friend.



Look at $E_\parallel$ above + below. Assume $\vec{\nabla} \times \vec{E} = 0$

Then Stokes thm $\Rightarrow \oint \vec{E} \cdot d\vec{l} = \int (\vec{\nabla} \times \vec{E}) \cdot d\vec{A} = 0$

And as loop gets small, must have $\boxed{E_{\parallel top} = E_{\parallel bot}}$

$E_\parallel$ always continuous across surfaces

Now, how is it $\vec{\nabla} \times \vec{E} = 0$? Easy way: Faraday's Law

$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ And nothing has time dependence.

Harder way: Straight from Coulomb's Law: $\quad E = \int \frac{K\rho(\vec{x}')\, d^3x' \, (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3}$

Use a trick: $\vec{\nabla} \frac{1}{r} = \frac{\vec{r}}{r^3}$ (see book, similar to saying $\frac{d}{dr}\frac{1}{r} = \frac{1}{r^2}$)

or $\quad \vec{\nabla} \frac{1}{|\vec{x} - \vec{x}'|} = \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3} \quad \Rightarrow \quad \vec{E} = \int \frac{K\rho(\vec{x}') (\vec{x} - \vec{x}')\, d^3x'}{|\vec{x} - \vec{x}'|^3}$

$$= \int K\rho(\vec{x}') \, \nabla \frac{1}{|\vec{x} - \vec{x}'|} \, d^3x'$$

∇ is w.r.t. real variable x, not dummy variable x', so

$$\vec{E} = \vec{\nabla} \int \frac{K\rho(\vec{x}')\, d^3x'}{|\vec{x} - \vec{x}'|}$$

$$\Rightarrow \vec{\nabla} \times \vec{E} = \vec{\nabla} \times \vec{\nabla}(\text{scalar}) = 0$$