EM plane waves

Since $\vec{E}$ interacts more strongly with electrons, normally keep track of $\vec{E}$ field

- calculate $\vec{B}$ from $\vec{E}$ (if necessary)
- with some waveguide configurations solve for B

Wave eqn from Maxwell

$$\nabla\cdot\vec{D} = 4\pi\rho \quad \rightarrow \cancel{\varepsilon}\nabla\cdot E = 0$$

assume medium is linear $\vec{D}=\varepsilon\vec{E}$, isotropic, no ρ_{free}

$$\nabla\cdot\vec{B}=0$$

$$\nabla\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$$

$$\nabla\times\vec{H} = \frac{1}{c}\frac{\partial\vec{D}}{\partial t} + \frac{4\pi}{c}J \quad\rightarrow\quad \nabla\times\vec{B} = \frac{\mu\varepsilon}{c}\frac{\partial\vec{E}}{\partial t}$$

$$\underbrace{\nabla\times(\nabla\times\vec{E})}_{} = -\frac{1}{c}\frac{\partial}{\partial t}\nabla\times\vec{B} = -\frac{\mu\varepsilon}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}$$

BAC-CAB ID:

$$\nabla(\cancel{\nabla\cdot\vec{E}}) - \nabla^2\vec{E}$$

0

Finally: $$\nabla^2\vec{E} = \frac{\mu\varepsilon}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}$$

3-D wave eqn
3 sets E_x, E_y, E_z

also: $$\nabla^2\vec{B} = \frac{\mu\varepsilon}{c^2}\frac{\partial^2\vec{B}}{\partial t}$$

specify geometry to get solutions.

- determines ∇^2 operator

$$\vec{E} = \hat{x}E_x(\vec{r},t) + \hat{y}E_y(\vec{r},t) + \hat{z}E_z(\vec{r},t)$$

cartesian, E_x component:

$$\nabla^2 E_x = \frac{\mu\varepsilon}{c^2}\frac{\partial^2 E_x}{\partial t^2}$$

<u>let $\mu \to 1$</u> (non-magnetic)

plane wave, $E_x(z,t)$ only

$$\frac{\partial^2 E_x}{\partial z^2} = \frac{\cancel{\mu}\varepsilon}{c^2}\frac{\partial^2 E_x}{\partial t^2}$$

solutions: $E_x = E_0 \cos(kz \pm \omega t + \phi)$

w/ $k = \frac{\omega}{c}\cdot\frac{1}{\sqrt{\varepsilon}}$

or separ. variables

$$E_x = f(z)g(t)$$

$$\frac{1}{f}\frac{\partial^2 f}{\partial z^2} = \frac{\varepsilon}{c^2}\frac{1}{g}\frac{\partial^2 g}{\partial t^2} = \text{const}$$

$$g = A_1 e^{i\omega t} + A_2 e^{-i\omega t}$$

$$\text{RHS} \to \frac{\varepsilon}{c^2}\cdot(-\omega^2)$$

$$f = B_1 e^{ikz} + B_2 e^{-ikz}$$

$$\text{LHS} \to -k^2$$

$$k^2 = \varepsilon\frac{\omega^2}{c^2}$$

<u>dispersion</u> relation

$\varepsilon = n^2$ $\quad n$ = refr. index

typ $\varepsilon = \varepsilon(\omega)$

$$E_x = f(z)g(t)$$

~~only 2 constants~~

E is real quantity

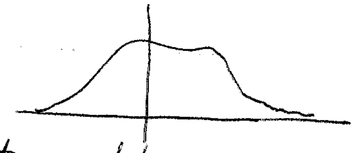
$$E_x = \underbrace{C_1 e^{i(kz-\omega t)} + C_2 e^{-i(kz-\omega t)}}_{\substack{\text{wave} \to +z \\ \text{forward}}} + \underbrace{C_3 e^{i(kz+\omega t)} + C_4 e^{-i(+)}}_{\substack{\to -z \\ \text{backward}}}$$

E real, absorb 2 const. on sin, cos to use amplitude + phase.

Wave theory – what is a wave?

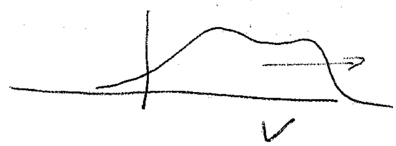
a wave propagates

consider a function $f(x)$



$f(x-a)$ is the same, shifted to right

$f(x+a)$ is shifted left.



$f(x-vt)$ moves to R

$f(x+vt)$ moves to L

1-D wave equation

$$\frac{\partial^2 f}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = 0$$

solution: $f(x \pm vt)$

let $u = x \pm vt$

$$\to \frac{\partial f}{\partial x} = \frac{du}{dx}\frac{df}{du} \qquad \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial u^2}$$

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial t}\right) = \pm v \frac{\partial}{\partial t}\frac{\partial f}{\partial u} \Rightarrow +v^2 \frac{\partial^2 f}{\partial u^2}$$

$$\therefore \frac{\partial^2 f}{\partial u^2} - \frac{1}{v^2} v^2 \frac{\partial^2 f}{\partial u^2} = 0 \quad \checkmark$$

- non trivial solution if $\partial^2 f/\partial u^2 \neq 0$
- $f(u)$ is the shape of wave in refer. frame comoving.

e.g. gaussian pulse

$$f(x,t) = e^{-(x-ct)^2/c^2\tau^2}$$

Notes

ω is not specified, can be anything. in vacuum
- $\varepsilon \rightarrow$ complex in general, absorption
  e.g. glass transmits visible, absorbs UV

general solution is any
$$f(x \pm vt)$$
Fourier then:
$$E = \sum A_i \cos(k(\omega_i) z \pm \omega_i t + \phi_i)$$

w/complex notation (phasors)
$$E(z,t) = \int_{-\infty}^{\infty} \underbrace{E(z,\omega)}_{\text{spectrum}} e^{i\omega t} \, d\omega$$

wave phase velocity:
$$V_{ph} = \frac{\omega}{k} \equiv \frac{k_0 c}{k_0 n} = \frac{c}{n} \qquad k_0 \equiv \frac{\omega}{c}$$

ω constant throughout system for light
λ varies
$\omega \equiv 2\pi\nu$ rad/s $\quad \nu$ in Hz
$k \equiv 2\pi/\lambda = 2\pi\sigma$ ↳ wave number (IR spectroscopy)

group velocity:
$$V_g = d\omega/dk \qquad \text{calculate } \boxed{\frac{1}{V_g} = \frac{d k(\omega)}{d\omega}}$$

dispersive medium:
$V_g \neq V_{ph}$ unless you work very hard.

Vector properties of EM waves

we normally keep track of E-field

- can get B-field from E-field.

plane wave $\vec{E}(x,y,z,t) = E_x(\vec{r},t)\hat{x} + E_y(\,)\hat{y} + E_z\hat{z}$

in general 3 components, each $f(\vec{r},t)$

Example $\vec{k} = k_0\hat{x}$ and $\vec{E}(\,)$ is a plane wave.

$\therefore$ no variation in y, z

$\vec{E}(x)$

$\nabla\cdot\vec{E} = 0 \rightarrow \frac{\partial E_x}{\partial x} = 0 \quad E_x = 0$

$\nabla\cdot\vec{B} = 0 \rightarrow \frac{\partial B_x}{\partial x} = 0 \quad B_x = 0$

E_x, B_x = constant is possible, but D.C. doesn't prop.

$\therefore$ plane wave is strictly transverse

- in isotropic, homogeneous medium

not in birefringent media, not for curved wavefronts.

skip

↓ Connecting $\vec{E}, \vec{B}$

simple example $\vec{k} = k_x\hat{x}$ $\vec{E} = E_y\hat{y}$

$\therefore \vec{E}(\vec{r},t) = \hat{y}E_y e^{i(k_x x - \omega t)}$

find $\vec{B}$ with:

$\nabla\times\vec{E} = -\frac{\partial B}{\partial t}$ $\qquad \nabla\times E = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 0 & E_y & 0 \end{vmatrix}$

$-\cancel{\partial_z E_y \hat{x}} + \partial_x E_y \hat{z} = -\frac{\partial \vec{B}}{\partial t}$ $\quad \therefore \vec{B}$ along $\hat{z}$

$$i k_x E_y e^{i(k_x x-\omega t)} = -\frac{\partial B_z}{\partial t}$$ dropping $\hat{z}$

$B_x = B_y = 0$

integrate

$$-i k_x E_y \int_0^t e^{i(k_x x-\omega t)}\, dt = B_z$$

$$\frac{-ik_x}{-i\omega} E_y \, e^{i(k_x x-\omega t)} = B_z$$

$$\omega = k_x c$$

$$\therefore \; B_z = \frac{1}{c} E_y$$ big? small?

- $\underline{\text{same}}$ phase
- $k \perp E \perp B$

force on electron

$$\vec{F} = q\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right)$$

w/ $B \approx E/c$

$$\vec{F} \approx q\left(\vec{E} + \frac{v}{c}E\right)$$

$\therefore$ down by v/c.

Generalization:

For plane waves $\quad \nabla\times\vec{E} \Rightarrow i\vec{k}\times\vec{E}$

$\nabla\cdot\vec{E} \to i\vec{k}\cdot\vec{E}$

ex. $(\nabla\times\vec{E})_x = \partial_y E_z - \partial_z \partial E_y = i(k_y E_z - k_z E_y)$

$= i(\vec{k}\times\vec{E})_x$

$\therefore \; i\vec{k}\times\vec{E} = -(-i\omega)\vec{B}$

$$\boxed{\begin{aligned}\vec{k}\times\vec{E} &= \frac{\omega}{c}\vec{B}\\ \vec{k}\cdot\vec{E} &= 0\end{aligned}}$$ for $\underline{\text{free plane waves}}$.