

Sampling + Signal recovery

- ubiquitous: digitized signals, numerical analysis

sampled function

$$f_s(t) = f(t) \operatorname{comb}\left(\frac{t}{t_s}\right)$$

multiply, not
convolve.

$$= \sum_n f(t) \delta(t - nt_s)$$

spectrum:

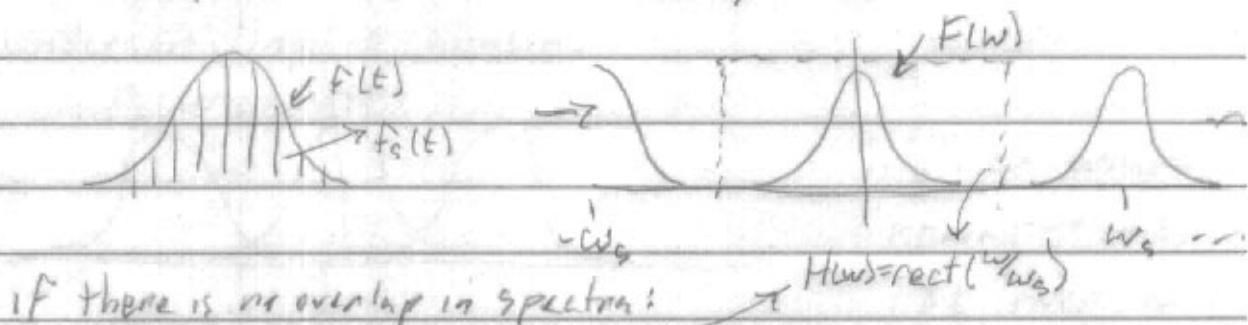
$$F_s(w) = \frac{1}{2\pi} F(w) \otimes \frac{2\pi}{t_s} \operatorname{comb}\left(\frac{w/2\pi}{2\pi/t_s}\right)$$

$$= \frac{1}{t_s} \sum_{n=-\infty}^{\infty} F\left(w - n \frac{2\pi}{t_s}\right)$$

$\hookrightarrow \equiv w_s = \text{sampling rate.}$

$F_s(w)$ is a continuous function

replicas of $F(w)$ spaced by w_s



If there is no overlap in spectrum:

- low-pass filter, recover exact original $F(w) \rightarrow f(t)$
- input signal must be BW limited
, if full width of input spectrum is W
 \rightarrow edge to edge.

Sampling rate $w_s \geq W$ $t_s \leq \frac{2\pi}{W}$

or max freq. is $W/2 = w_{max} \rightarrow$ period $T_{min} = \frac{2\pi}{w_{max}}$

$t_s \leq \frac{1}{2} T_{min}$ Nyquist limit

note that many filters are characterized by their $\frac{1}{2}$ power width

i.e. higher frequencies get through.

→ "oversampling"

Signal recovery:

$$g(\omega) = H(\omega) F_g(\omega)$$

$$= \text{rect}\left[\frac{\omega/\omega_s}{2}\right] \cdot \frac{1}{t_0} \sum_n F(\omega - n\omega_s)$$

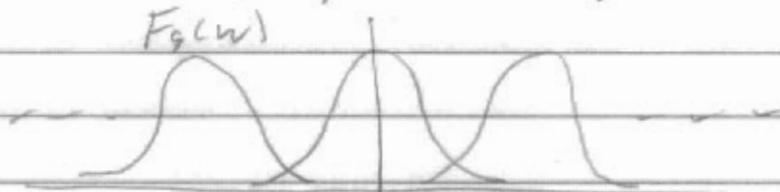
alternatively,

$$g(t) = h(t) \otimes f_s(t)$$

$$= \frac{\omega_s}{2\pi} \text{sinc}\left(\frac{\omega_s}{2}t\right) \otimes f_s(t)$$

so the sinc acts as the perfect interpolating function.

Undersampling → aliasing



high ω wraps around → looks like lower ω .

Limited sampling duration:

- clips input

$$f'_s(t) = f(t) \cdot \text{comb}\left(\frac{t}{t_0}\right) \cdot \text{rect}\left(\frac{t}{T}\right)$$

$$F'_s(\omega) = \frac{1}{2\pi} F_s(\omega) \otimes T \text{sinc}\left(\frac{\omega T}{2}\right)$$

★ spectral resolution is smeared out by finite T

finite bin width (pixel size)

- instead of comb, integrate over a box



$$\hat{f}_s(nt_s) = \frac{1}{b} \int_{-\infty}^{\infty} f(t) \operatorname{rect}\left(\frac{t-nt_s}{b}\right) dt'$$

we can think of $nt_s \rightarrow t$ (continuous)

so that digitized signal is

$$f_d(t) = \hat{f}(t) \operatorname{comb}\left(\frac{t}{t_s}\right)$$

$$\begin{aligned} \hat{f}(t) &= \frac{1}{b} \int f(t') \operatorname{rect}\left(\frac{t'-t}{b}\right) dt' \\ &= f(t) \otimes \underbrace{\operatorname{rect}\left(\frac{t}{b}\right)}_{\text{symmetric lemma}} \end{aligned}$$

this has the effect of low-pass on the input
digitized spectrum is

$$\hat{F}_d(w) = \left[F(w) : \operatorname{sinc}\left(\frac{w \cdot b}{2}\right) \right] \otimes \operatorname{comb}\left(\frac{w}{w_s}\right)$$

but $\operatorname{sinc}(\cdot)$ never truly goes to zero,
might be able to divide it out

$$H(w) = \frac{1}{\operatorname{sinc}\left(\frac{w \cdot b}{2}\right)} \operatorname{rect}\left(\frac{w}{w_s}\right)$$

FFT - fast version of the discrete FT
computation time $\propto N^{\frac{1}{2}}$ in one dimension.

using FFT

1) Sample $f(t)$ $t_s = \delta t$

At $s_0 t$'s maximum bandwidth

full width $\omega_{FWH} = 2\pi/\delta t$

$$\text{so } \delta \omega = \frac{2\pi}{N\delta t}$$

use this to determine areas in t, ω space.

Normalization:

$$\text{want } \sum |E_i|^2 \delta t = \sum |\tilde{E}_i|^2 \delta \omega$$

FFT on its own preserves energy, in the sense
that $\sum |F_i|^2 = \sum |f_i|^2$

we must define pulse energy as

$$E = \frac{1}{2} \epsilon_0 c n \sum |E_i|^2 \delta t$$

this way E is independent of sampling rate.

In ω domain, we want

$$E = \frac{1}{2} \epsilon_0 n \sum |\tilde{E}_i|^2 \delta \omega$$

\therefore define $\tilde{E} = \left(\frac{\delta t}{\delta \omega} \right)^{\frac{1}{n}} \text{Fourier}[E]$

$$\left(\frac{N \delta t^2}{2\pi} \right)^{\frac{1}{n}}$$

Pulse propagation through dispersive material

can view this in frequency and time domains.

review of classical dispersion theory:

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

→ induced polarization

many interrelationships:

$$\text{gas. } \vec{P} = N_a \vec{p} \quad N_a: \# \text{ density of atoms}$$

\vec{p} = dipole moment of individual atom

$$\vec{P}(t) = -e \vec{x}(t)$$



mean position of e moves in response to \vec{E}
write force eqn:

$$m\ddot{x} + m\gamma\dot{x} + Kx = F(t) = q\vec{E}(t)$$

2 ways: assume all goes as $e^{-i\omega t}$

$$\Rightarrow -m\omega^2 x_0 - i\omega m\gamma x_0 + Kx_0 = qE_0$$

where x_0 is an amplitude (complex)

say $E_0 = \alpha$, $x_0 = \beta$

$$\rightarrow \omega^2 = \frac{K}{m} \rightarrow \text{resonant frequency } \omega_0$$

$$K = m\omega_0^2$$

$$x_0 = \frac{(\gamma/m)E_0}{(\omega_0^2 - \omega^2) - i\omega\gamma} \quad \text{frequency dependent.}$$

atomic polarizability is

$$\alpha(\omega) = \frac{P}{\epsilon_0 E_0} = \frac{q^2}{\epsilon_0 m (\omega_0^2 - \omega^2 - i\omega\gamma)}$$

Second way: $\int \{ \}$ on both sides.

$$\int (\partial f / \partial t) e^{-i\omega t} dt = -i\omega F(\omega)$$

$$\rightarrow -m\omega^2 \tilde{X} - i m \gamma \omega \tilde{X} + m \omega_0^2 \tilde{X} = q \tilde{E}$$

same result.

Point is that we're really solving eqn. in freq. domain

calculate index of refraction:

$$P = N_a p = N_a q X = \frac{N_a q^2}{m(\omega_0^2 - \omega^2 - i\gamma\omega)} E$$

linear response

$$\vec{P} = \epsilon_0 \chi^{(1)} \vec{E}$$

↙
susceptibility

ω_0 may vary with direction of \vec{E}

→ tensor for $\chi^{(1)}$, birefringence

\vec{P} may have non-linear response

$$\rightarrow \chi^{(1)} + \frac{1}{2} \vec{E} \chi^{(2)} + \frac{1}{3!} \vec{E} \cdot \vec{E} \chi^{(3)}$$

$$D = \epsilon_0 E + P = \underbrace{\epsilon_0 (1 + \chi^{(1)})}_{\epsilon_r} E$$

ϵ_r = relative permittivity = n^2

$$n^2 = 1 + \frac{N_a q^2}{\epsilon_0 m (\omega_0^2 - \omega^2 - i\gamma\omega)} = r(\omega), \text{ complex}$$

$\text{Im}(n) \rightarrow$ absorption

inside condensed media: $\frac{\epsilon_r - 1}{\epsilon_r + 2} = \frac{2}{3} \epsilon_r \chi^{(1)}$ not sum of $\epsilon_r, 2\pi$ factors
 real materials: empirical eqns for new.
 see Hecht, Marion
 19.3

Pulse propagation:

use frequency domain to apply correct modification to wave.

after propagation from $z=0$ to L

$$\tilde{E}_0(L, \omega) = \tilde{E}_0(0, \omega) e^{ik_0 n(\omega)L}$$

so in time domain,

$$E(L, t) = \mathcal{F}^{-1} \left\{ \tilde{E}(0, \omega) e^{ik_0 n(\omega)L} \right\}$$

From here:

ω -domain - notice $k_0 n(\omega)L = \frac{\omega n(\omega)L}{c} = \phi(\omega)$

→ spectral phase

- expand $\phi(\omega) = \phi(\omega_0) + \phi'(\omega_0)\Delta\omega + \frac{1}{2}\phi''(\omega_0)\Delta\omega^2 + \text{HOT.}$

- invert to time domain

t -domain - diff' kinds of chirp

- impulse response of dielectrics

Spectral phase.

represent input pulse in ω -domain

$$\tilde{E}(\omega, z) = E_0 e^{ik_0 z} \frac{e^{-(\omega - \omega_0)^2 \tau^2 / 4}}{\sqrt{\pi} \tau}$$

$k_0 z$ is propagation phase in vacuum

optical material



$n(\omega)$ = index of refr., may be complex

input $\tilde{E}(\omega, 0)$

$$\text{output } \tilde{E}(\omega, L) = \tilde{E}(\omega, 0) e^{-i \frac{\omega n(\omega)}{c} L}$$

transfer function for "system" is

$$H(\omega) = e^{i \frac{\omega n(\omega)}{c} L} = e^{-\frac{i \omega n(\omega)}{c} L} e^{i \phi(\omega)}$$

amp block
less loss/gain $\phi(\omega) = \text{spectral phase.}$

consider transformation mat'l first $T_m(n) \approx 0$

$\left. \frac{d}{d\omega} \right|_{\omega_0}$ is hard in general

1) \rightarrow numerate (FPT)

2) \rightarrow Taylor expand.

$$\begin{aligned} \left. \phi(\omega) \right|_{\omega_0} &= \phi(\omega_0) + (\omega - \omega_0) \phi'(\omega_0) + \frac{1}{2} (\omega - \omega_0) \phi''(\omega_0) \\ &\quad + \frac{1}{3!} (\omega - \omega_0)^3 \phi'''(\omega_0) + \dots \end{aligned}$$

for now, keep thru 2nd order.

$$E(t, L) = \frac{1}{2\pi} \sqrt{\pi} T^2 E_0 \int d\omega \exp \left[-\frac{\Delta\omega^2 T^2}{4} + i\phi_0 + i\Delta\omega \phi'_0 + \frac{i\Delta\omega^2 \phi''_0 - i\omega t}{2} \right]$$

$$= \frac{T}{\sqrt{4\pi}} E_0 e^{i\phi_0 - i\omega_0 t} \int d(\Delta\omega) \exp \left[-\left\{ \left(\frac{T^2}{4} - \frac{i\phi''_0}{2} \right) \Delta\omega^2 + i(t - \phi'_0) \Delta\omega \right\} \right]$$

retain: constant phase pulls out: $i\omega_0 \left(n(\omega_0) \frac{L}{c} - \phi'_0 - T \right)$
 $\ell \frac{i(n(\omega_0) L - \omega_0 t)}{c}$

time variable is shifted $T = t - \phi'_0$

$$\phi'_0 = \left. \frac{d\phi}{d\omega} \right|_{\omega_0} = \text{group delay} \quad \text{if } k = \frac{\omega_0 n(\omega_0)}{c}$$

$$\frac{dk}{d\omega} = \frac{1}{v_g}$$

$$\tau_g = \frac{L}{v_g}$$

let $(\tau')^2 = 4 \left(\frac{T^2}{4} - \frac{i\phi''_0}{2} \right)$ this will be new pulse duration.

complete square

$$\frac{1}{4} (\tau')^2 \left[\left(\Delta\omega - \frac{iT}{(\tau')^2/2} \right)^2 + \frac{4T^2}{(\tau')^4} \right]$$

$$\text{let } u^2 = \frac{(\tau')^2}{4} \left(\Delta\omega - \frac{iT}{(\tau')^2/2} \right)^2$$

$$du = \frac{\tau'}{2} d(\Delta\omega)$$

$$\int e^{-u^2} du = \sqrt{\pi}$$

$$E(t, L) = \frac{E_0}{\tau'} \frac{2\sqrt{\pi}}{\sqrt{4\pi}} e^{i(\phi_0 - \omega_0 t)} e^{-T^2/(\tau')^2}$$

$$\frac{1}{(\tau')^2} = \frac{1}{\tau'^2 - 2i\phi''_0} = \frac{\tau'^2 + 2i\phi''_0}{\tau'^4 + 4(\phi''_0)^2} = \frac{1 + i^2 \phi''_0 / \tau'^2}{\tau'^2 (1 + (2\phi''_0 / \tau'^2)^2)}$$

$$\text{let } b = 2\Phi_0''/\tau^2 \quad \Phi_0'' = \frac{L}{C} \frac{d^2}{dw^2} (\omega n(w)) \Big|_{w_0}$$

$$\tau_c = \tau \sqrt{1+b^2}$$

$$\frac{\tau}{\tau'} \rightarrow \frac{\tau}{\tau_c} \sqrt{1+b^2} \rightarrow \frac{1}{(1+b^2)^{1/4}} e^{i \tan^{-1}(b)}$$

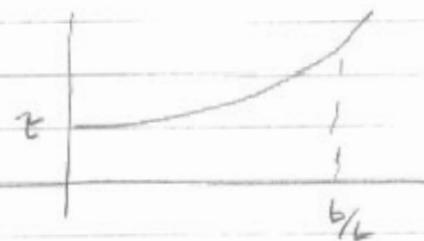
$$E(t, L) = \frac{E_0}{(1+b^2)^{1/4}} e^{i\phi_0} e^{-i\omega_0 T} e^{-ibT^2} = T^2/\tau_c^2$$

$$\Phi_t = \Phi_0 - \omega_0 \phi_0' + \tan^{-1} b \quad \text{all const. phase shift.}$$

since $I \sim |E|^2$ peak intensity falls as pulse stretches.

$$\tau/\tau_c$$

pulse duration is $\Gamma(L)$



$$bT^2 = \text{temporal phase} = \phi(T)$$

T is now local time in ref. frame moving w/pulse.

note - ωT is a phase $\rightarrow -\frac{d}{dT}(-\omega T) \rightarrow \text{freq.}$

i.e. $-\frac{d}{dT} \phi(T) = \text{instantaneous frequency}$

here $= -2bT$ linear chirp

$$\text{Compare } \phi(w) = \phi_0 + \Delta\omega \phi_0' + \frac{1}{2}(\Delta\omega)^2 \phi_0''$$

$$\text{group delay } \tau_g = \frac{d\phi}{d\omega} = \phi_0' + \Delta\omega \phi_0'' \quad \text{also linear in } \omega$$

$$\tau_g(w) = \text{arrival time as } f(w)$$

Varying group delay is what controls shape of pulse.

$$\text{I.R. } \mathcal{E}_g(w) - \phi'$$

higher orders $\phi'' \rightarrow$ parabolic chirp



"fringes" or "ringing" at leading or trailing edge
- fringe interference/beating of opposite ends of spectrum.

"Absolute" phase

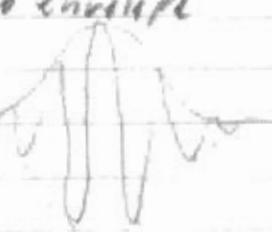
T is relative to peak of pulse.

$$i(\phi_L - w_0 T)$$

e

if $\phi_L = 0$ carrier would always be fixed in phase relative to envelope

$\phi_L \neq 0$ carrier-envelope phase changes.



$$\phi_L = \phi_0 - w_0 \phi'_0 + \tan^{-1} b$$

$$= \frac{w_0 n(w_0)}{c} L - w_0 \left. \frac{\partial}{\partial w} (n(w)) \right|_{w=w_0} \left| \frac{L}{c} \right| + \tan^{-1} \left(2 \frac{\phi''(w_0)}{c^2} \right)$$

$$= w_0 L \left(\frac{1}{v_{ph}} - \frac{1}{v_{ge}} \right) + \tan^{-1} ()$$

phase - group vel mismatch.

it is possible to stabilize this in a laser oscillator.