MATH 225 - Differential Equations June 16 , 2008

Homework 9, Field 2008 **Due Date:** June 17, 2008

STABLE COMPETITION - COMPLETIVE EXCLUSION

Today in class we studied the nonlinear system of differential equations,

$$\frac{dx}{dt} = x(10 - x - y) \quad (1)$$

$$\frac{dy}{dt} = y(30 - 2x - y), \quad (2)$$

and found out that there existed four equilibrium points and that the two-species physical solutions (solutions in the first quadrant of phase space) tended to the a real sink at $(x, y) = (0, 30)$. We say that this competition is unstable in the sense that in competitive situations all x trajectories tend to zero as t goes to infinity.

Suppose now that we have the following system of differential equations,

$$\frac{dx}{dt} = x(2 - 2x - y) \quad (3)$$

$$\frac{dy}{dt} = y(2 - x - 2y), \quad (4)$$

where $x, y \in [0, \infty)$, which models competition in two species x and y.

1. Find all fixed points of the system.

2. Using the Jacobian matrix classify these fixed points.

3. Do a full eigenvalue/eigenvector decomposition of the off-axis equilibrium and using this approximate solution discuss the local behavior of the system in a neighborhood of this equilibrium.

4. Using HPGSYSTEMSOLVER plot the slope field in the first quadrant and plot enough trajectories to give a qualitative description of the time-dynamics of the populations. Is the system stable in the sense that there exists an equilibrium population such that both x and y are nonzero?

5. Using HPGSYSTEMSOLVER plot the slope field of the system of differential equations,

$$\frac{dx}{dt} = x(2 - x - 2y) \quad (5)$$

$$\frac{dy}{dt} = y(2 - 2x - y), \quad (6)$$

where $x, y \in [0, \infty)$, which models competition in two species x and y. Compare the time dynamics of this model to the model (3)-(4) and discuss the differences in stability.

Stable Competition/Comp. Exclusion
Borrelli + Coleman 1st ed.
pg 282-283

1)

$$(1)\quad \frac{dx}{dt} = x(2-2x-y) = f(x,y)$$
$$\frac{dy}{dt} = y(2-x-2y) = g(x,y)$$

a) Equilibrium Points

$P_1 = (0,0)$

$P_2 = (1,0)$

$P_3 = (0,1)$

$P_4 = (2/3, 2/3)$

Work for P_4.

$2-2x-y=0 \Rightarrow y = 2-2x$

$2-x-2y=0 \Rightarrow 2-x-2(2-2x) = 2-x-4+4x =$

$= -2+3x = 0$

$x = \frac{2}{3} \Rightarrow 2-2x-y =$

$= 2-\frac{4}{3}-y=0$

$\Rightarrow y = 2/3$

b) $$J(x,y) = \begin{bmatrix} f_x(x,y) & f_y(x,y) \\ g_x(x,y) & g_y(x,y) \end{bmatrix} =$$

$$= \begin{bmatrix} 2-4x-y & -x \\ -y & 2-4y-x \end{bmatrix}$$

$P_1(0,0) \Rightarrow J(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow \lambda = 2$ (Repeated Source)

$P_2 = (1,0) \Rightarrow J(1,0) = \begin{bmatrix} -2 & -1 \\ 0 & 1 \end{bmatrix} \Rightarrow \lambda_1 = -2, \lambda_1 = 1 \Rightarrow$ Saddle Node

$P_3 = (0,1) \Rightarrow J(0,1) = \begin{bmatrix} 1 & 0 \\ -1 & -2 \end{bmatrix} \Rightarrow \lambda_1 = 1, \lambda_2 = -2 \Rightarrow$ Saddle Node

$P_4 = (2/3, 2/3) \Rightarrow J(2/3, 2/3) = \begin{bmatrix} -4/3 & -2/3 \\ -2/3 & -4/3 \end{bmatrix} \Rightarrow T = -8/3, D = 4/3 \Rightarrow$ Some Sort of Sink.

c) At point P_4 we have the linear system

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4/3 & -2/3 \\ -2/3 & -4/3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \qquad (2)$$

which approximates the nonlinear system at P_4.

$$\det(A-\lambda I) = \det\left(\begin{bmatrix} -4/3-\lambda & -2/3 \\ -2/3 & -4/3-\lambda \end{bmatrix}\right) =$$

$$= \left(\frac{-4}{3}-\lambda\right)^2 + \frac{-4}{9} = 0$$

$$\Rightarrow \quad -\frac{4}{3}-\lambda = \pm\sqrt{\frac{4}{9}} = \pm\frac{2}{3}$$

$$\Rightarrow \quad -\lambda = \pm\frac{2}{3}+\frac{4}{3} \quad \Rightarrow \quad \lambda_1 = -\frac{6}{3} = -2$$

$$\lambda_2 = -\frac{2}{3}$$

Case $\lambda_1 = -2$

$$(A-\lambda_1 I)\vec{v} = \begin{bmatrix} -4/3+\overset{6/3}{2} & -2/3 \\ -2/3 & -4/3+2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \frac{2}{3}v_1 - \frac{2}{3}v_2 = 0 \quad \Rightarrow \quad v_1 = v_2 \quad \Rightarrow \quad \vec{v}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Case $\lambda_2 = -\frac{2}{3}$

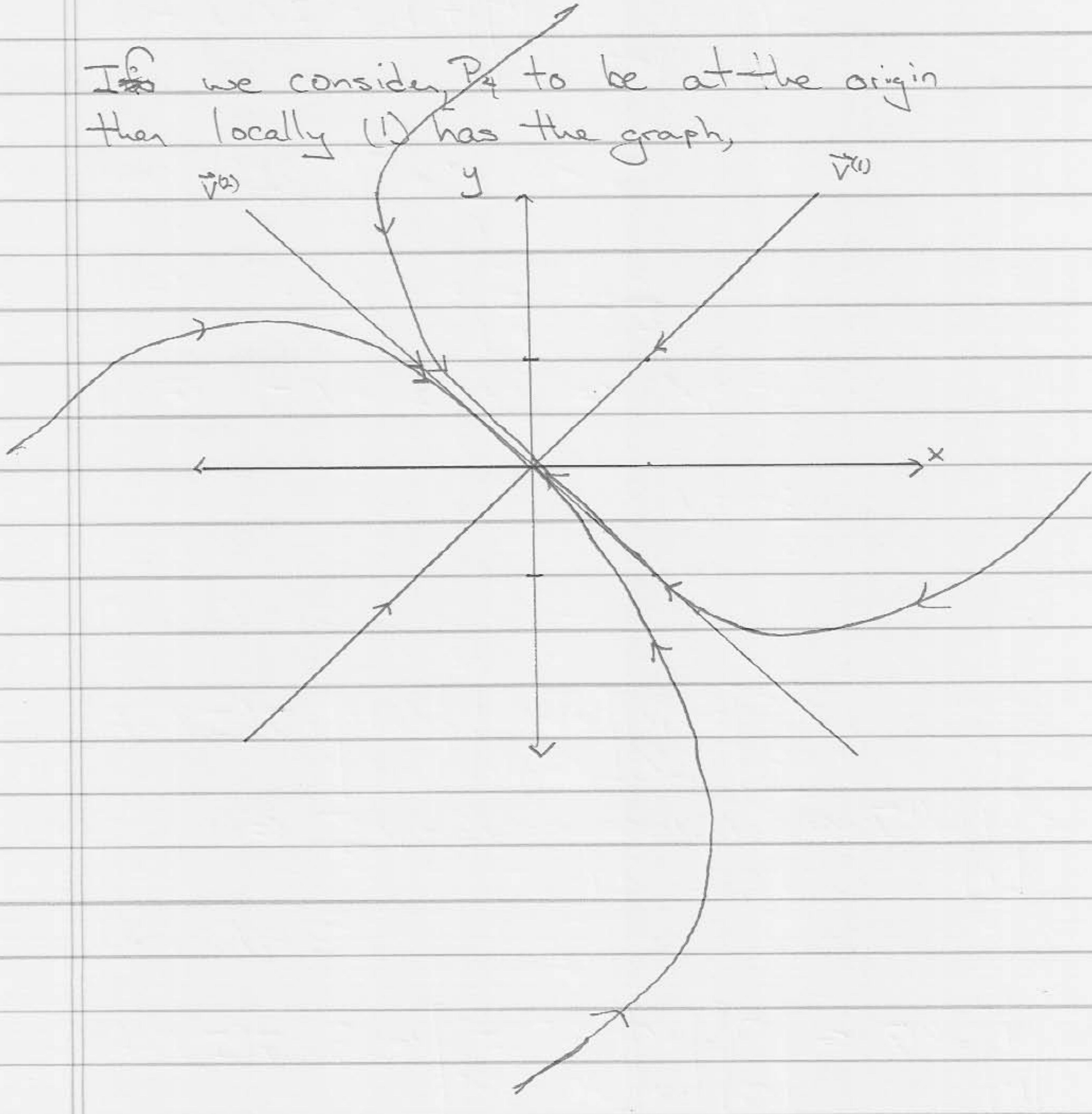
$$(A-\lambda_2 I)\vec{v} = \vec{0} \quad \Rightarrow \begin{bmatrix} -\frac{2}{3} & -2/3 \\ -2/3 & -2/3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \vec{v}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

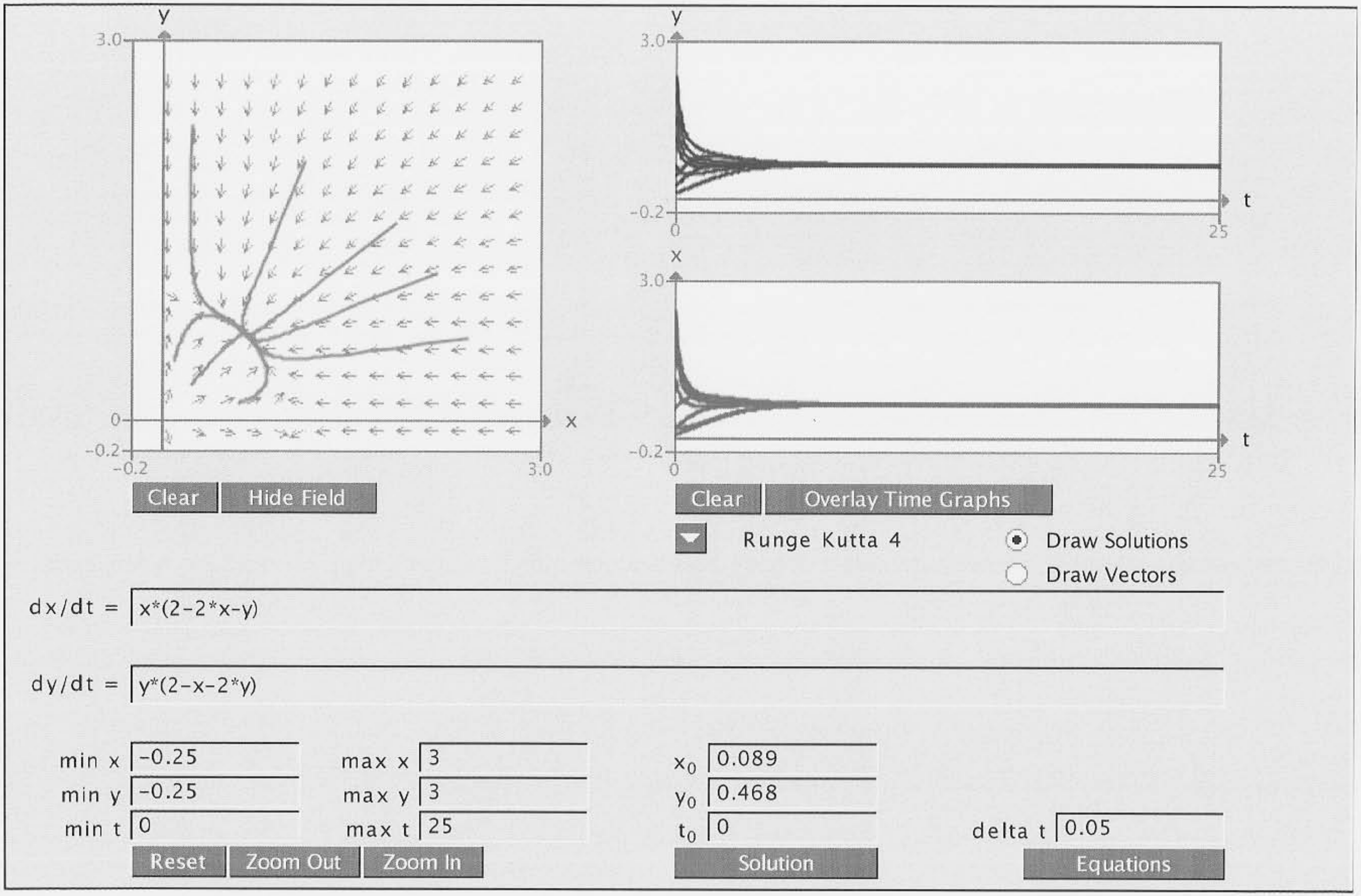
Thus, the general soln to (2) is given by,

$$\vec{Y}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2/3 t}$$

which implies that (1) is locally like a Real sink at P_4.

If we consider P_4 to be at the origin then locally (1) has the graph,





y
3.0
0
-0.2
-0.2
3.0
x
Clear
Hide Field
y
3.0
-0.2
0
25
t
x
3.0
-0.2
0
25
t
Clear
Overlay Time Graphs
Runge Kutta 4
Draw Solutions
Draw Vectors
dx/dt = x*(2-2*x-y)
dy/dt = y*(2-x-2*y)
min x -0.25
max x 3
min y -0.25
max y 3
min t 0
max t 25
x0 0.089
y0 0.468
t0 0
delta t 0.05
Reset
Zoom Out
Zoom In
Solution
Equations

d) Based on (c) and the graph we see that there is an equilibrium population, P_4, which $\text{sol}^{\underline{n}}$ tend to for large t and that this population has nonzero x and y. Thus, there is a stable long term coexisting pop.

e) Consider now,

$$\frac{dx}{dt} = x(2-x-2y) = 2x - x^2 - 2xy$$

$$\frac{dy}{dt} = y(2-2x-y) = 2y - 2xy - y^2$$

$P_1 = (0,0)$ $\qquad 2-x-2y=0 \Rightarrow y = 1 - \frac{x}{2}$

$P_2 = (2,0)$

$P_3 = (0,2)$ $\qquad 2-2x-y = 2-2x-\left(1-\frac{x}{2}\right) = 1 - \frac{3}{2}x = 0$

$P_4 = (2/3, 2/3)$ $\qquad \Rightarrow x = \frac{2}{3}$

$y = \frac{2}{3}$

$$J(x,y) = \begin{bmatrix} 2-2x-2y & -2x \\ -2y & 2-2x-2y \end{bmatrix}$$

$J(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow P_1$ is a Real Rep Source

$J(2,0) = \begin{bmatrix} -2 & -4 \\ 0 & -2 \end{bmatrix} \Rightarrow P_2$ is a $\overset{\text{Real}}{\text{Sink}}$

$J(0,2) = \begin{bmatrix} -2 & 0 \\ -2 & -2 \end{bmatrix} \Rightarrow P_3$ is a Real Sink

$$J(2/3, 2/3) = \begin{bmatrix} -2/3 & -4/3 \\ -4/3 & -2/3 \end{bmatrix}$$

which gives,

$$\det(J - \lambda I) = \left(-\frac{2}{3} - \lambda\right)^2 - \frac{16}{9} = 0$$

$$\Rightarrow -\lambda - \frac{2}{3} = \pm\frac{4}{3} \Rightarrow -\lambda = \pm\frac{4}{3} + \frac{2}{3}$$

$$\lambda_1 = +\frac{2}{3} \quad \lambda_2 = -2$$

Case $\lambda_1 = +2/3$

$$(J - \lambda_1 I)\vec{v} = \begin{bmatrix} -4/3 & -4/3 \\ -4/3 & -4/3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \vec{v}^{(1)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Case $\lambda_2 = -2$

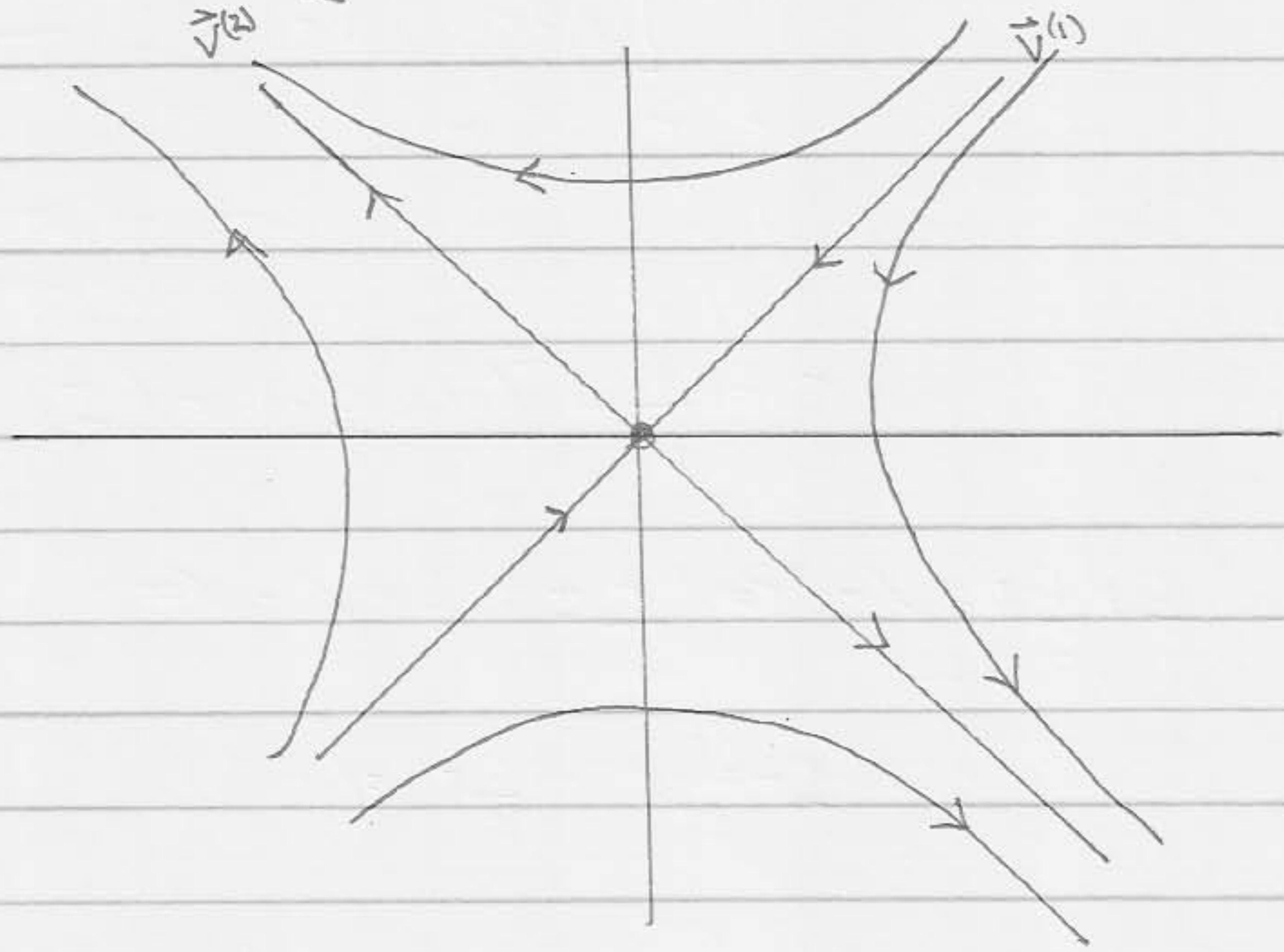
$$(J - \lambda_2 I)\vec{v} = \vec{0} \Rightarrow \begin{bmatrix} -2/3 + 6/3 & -4/3 \\ -4/3 & -2/3 + 6/3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \frac{4}{3}v_1 - \frac{4}{3}v_2 = 0 \Rightarrow \vec{v}^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Thus, the soln to the linearized system at P_4 is given by,

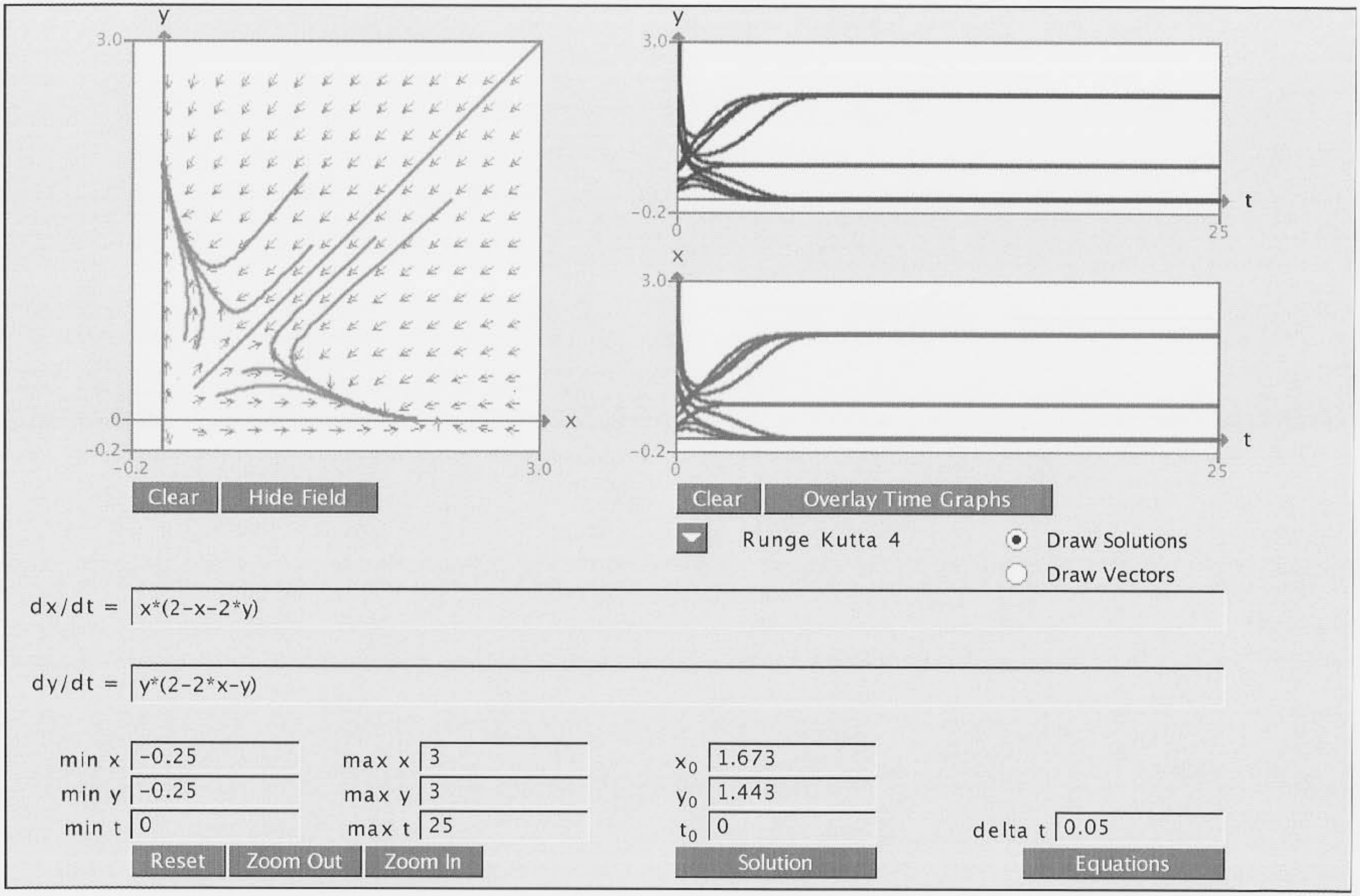
$$\vec{Y}(t) = C_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{-2t} + C_2\begin{bmatrix} -1 \\ 1 \end{bmatrix}e^{+2/3\,t}$$

and locally the nonlinear system looks like



which is a saddle node.

In this case, see next page for the figure, there does not exist a ~~sab~~ stable pop. which <u>all</u> soln trajectories go to for large t. If the initial conditions are such that the traj. starts on $\vec{v}^{(1)}$ then the pops will go to P_4. Otherwise they will diverge from P_4 to P_2 or P_3.


y
3.0
0
-0.2
-0.2
3.0
x
Clear
Hide Field
y
3.0
-0.2
0
25
t
x
3.0
-0.2
0
25
t
Clear
Overlay Time Graphs
Runge Kutta 4
Draw Solutions
Draw Vectors
dx/dt = x*(2-x-2*y)
dy/dt = y*(2-2*x-y)
min x -0.25
min y -0.25
min t 0
max x 3
max y 3
max t 25
x_0 1.673
y_0 1.443
t_0 0
delta t 0.05
Reset
Zoom Out
Zoom In
Solution
Equations