Physics 350 – Undergraduate Classical Mechanics
Numerical Homework III, due Friday, October 21 at 11:00 a.m.

All solutions must be in hard-copy form; no electronic copies will be accepted.
All solutions must be typed.

Now that we know how to use Lagrangian mechanics to formulate differential equations for complicated situations, let's apply it to one of the classical systems that demonstrates chaos: the double pendulum.

Here is the problem. You have a double pendulum just like you solved for the Lagrangian in the last analytical homework (Thornton problem 7.7). Set up a numeric solver that can take the masses and lengths of the arms of the pendulum as well as initial conditions (θ, ϕ, $d\theta/dt$, $d\phi/dt$). For this assignment, we'll take the length of both arms of the pendulum to 0.25m, and the masses of both bobs on the end of the pendulum arms to be 3kg. Use $g=9.8m/s^2$.

1) You can either solve for the second time derivative of the angles using your analytic homework, or you can use Mathematica to rederive the Lagrange equations of motion and solve for them in terms of the generalized coordinates and velocities. Get expressions of these that you will use to give you the generalized accelerations at each time step.

2) Make a module that will calculate the two angles as functions of times. I would make the inputs for the modules include the maximum time, the time step, and the initial angles and angular velocities. It's up to you if you want to include more than that. Note that while I was doing this, I found that you can speed up your module executions by more than a factor of 10 by using the Compile function offered in Mathematica. I would definitely recommend using this, especially if you are going to do the extra credit part of the assignment.

3) Test your results by seeing how much energy conservation is violated for given time steps. Also animate the motion of your pendulum. Outside of looking really cool, you can typically tell if something isn't right by visualizing the motion. Between the energy conservation and the visualization, you should be able to hunt down errors. Here is some code that you can use to plot and animate your pendulum:

```
posViewer[th1_, ph1_, params_] := (
  Clear[t];
  vpos1 = pos1 /. {th[t] → th1, ph[t] → ph1} /. params;
  vpos2 = pos2 /. {th[t] → th1, ph[t] → ph1} /. params;
  Graphics1 = {Red, Disk[vpos1, 0.03]};
  Graphics2 = {Blue, Disk[vpos2, 0.03]};
  GraphicsLine1 = {Thick, Black, Line[{{0, 0}, vpos1, vpos2}]};
  scale = 1.2;
  Graphics[Flatten[{GraphicsLine1, Graphics1, Graphics2}],
   PlotRange → ({{- (L1 + L2) * scale, (L1 + L2) * scale},
       {- (L1 + L2) * scale, (L1 + L2) * scale}} /. params)]
 )
Animate[posViewer[thArr[[i, 2]], phArr[[i, 2]], pParams], {i, 1, Length[thArr], 1}]
```

where I had defined pos1 and pos2 earlier in the code with {x coord as a function of theta and phi, y coord as a function of theta and phi}

4) Stuff we'll grade you on:
   a. Error in energy using a dt of 0.001 seconds and maximum time of 10 seconds:
      i. Plot the % error in energy as a function of time for the following initial conditions: θ = 10 deg, ϕ=20 deg, dθ/dt = 0, dϕ/dt = 0
      ii. Print out the % error in energy density for the following cases:
         1. θ = 10 deg, ϕ=20 deg, dθ/dt = 0, dϕ/dt = 0 (~0.5%)
         2. θ = 30 deg, ϕ=60 deg, dθ/dt = 0, dϕ/dt = 0 (~5%)
         3. θ = 90 deg, ϕ=180 deg, dθ/dt = 0, dϕ/dt = 0 (~15%)
         4. θ = 0 deg, ϕ=0 deg, dθ/dt =0, dϕ/dt = 50 deg/s (~0.08%)
         5. θ = 0 deg, ϕ=0 deg, dθ/dt =0, dϕ/dt = 100 deg/s (~0.3%)
         6. θ = 0 deg, ϕ=0 deg, dθ/dt =0, dϕ/dt = 500 deg/s (~8%)
   b. Run animations of your pendulum for the cases 1 and 3 from part (a). Make a snapshot of the animation from 3 as proof (the animation updates as you run stuff after it.
   c. Do a convergence study. One way to do this is to plot –log(dt) vs the change in final angle going from one dt to the next. Change dt from 0.01 to 0.005 to 0.001 to 0.0005 to 0.0001 and plot –log(dt) on the horizontal axis and log(change in final θ going from the last dt to the current one/current final θ) on the vertical axis. You should get something that gets smaller and smaller as –log(dt) gets larger (dt gets smaller). For this, to make it solve faster, make your maximum time only 1 second.
   d. Plot θ as a function of time for the following cases using a maximum time of 10 seconds and a dt of 0.001 seconds. They should all be plotted on the same axis, so one graph with four curves on it.
      i. θ = 20 deg, ϕ=40 deg, dθ/dt = 0, dϕ/dt = 0
      ii. θ = 20 deg, ϕ=40.01 deg, dθ/dt = 0, dϕ/dt = 0
      iii. θ = 20 deg, ϕ=40.02 deg, dθ/dt = 0, dϕ/dt = 0
      iv. θ = 20 deg, ϕ=40.03 deg, dθ/dt = 0, dϕ/dt = 0

e. Plot θ as a function of time for the following cases using a maximum time of 10 seconds and a dt of 0.001 seconds. They should all be plotted on the same axis, so one graph with four curves on it.
    i. θ = 90 deg, ϕ=180 deg, $d\theta/dt = 0$, $d\phi/dt = 0$
    ii. θ = 90 deg, ϕ=180.01 deg, $d\theta/dt = 0$, $d\phi/dt = 0$
    iii. θ = 90 deg, ϕ=180.02 deg, $d\theta/dt = 0$, $d\phi/dt = 0$
    iv. θ = 90 deg, ϕ=180.03 deg, $d\theta/dt = 0$, $d\phi/dt = 0$

f. Extra credit: This could take all night to run depending on how you program it, so don't plan on doing this at the last minute. I want you to plot the final θ using the maximum time of 10 seconds and a dt of 0.001 seconds. Do this as a function of starting θ (final angle will go on the vertical axis and initial will go on the horizontal axis of the plot for this). Make all the initial angular velocities zero, and set the initial ϕ to twice the initial θ. Here's how to make the plot. Start with the initial θ=0. The final will obviously be 0. Now perturb it by 0.001 degrees. So θ = 0.001 deg, ϕ=0.002 deg. Do this 20 times (keep adding on 0.001 each time). Plot each point (θ_i,θ_f) on the graph. You just did a stability study for those initial conditions. Now do that for θ_i ranging from 0 to 90 degrees keeping ϕ_i equal to twice θ_i, and the velocities zero. I would change θ_i in steps of 5 degrees. Do 20 perturbations each of 0.001degrees from the previous one. Plot all these points, and comment on what you see.