

Matrix Algebra - Row Reduction - Solutions to Linear Systems

1. We know, by counterexamples, that matrix multiplication is a non-commutative binary operation. We define a commutator as a function, which takes in two matrices and returns one and is, in some sense, a measure of the binary operations lack of commutativity. We define the commutator and anti-commutation functions on matrices as,

$$[A, B] = AB - BA, \quad \{A, B\} = AB + BA. \quad (1)$$

The following matrices are the so-called Pauli spin matrices and have interesting commutation and anti-commutation relations and gives us fine setting to practice our matrix algebra.¹

$$\sigma_1 = \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (2)$$

Using the previous definitions show the following:

- (a) $\sigma_i^2 = \mathbf{I}$ for $i = 1, 2, 3$.²
- (b) $[\sigma_i, \sigma_j] = 2i \sum_{k=1}^3 \epsilon_{ijk} \sigma_k$ for $i = 1, 2, 3$ and $j = 1, 2, 3$.³
- (c) $\{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbf{I}$ for $i = 1, 2, 3$ and $j = 1, 2, 3$.⁴

2. Given the linear system

$$\begin{aligned} 6x_1 + 18x_2 - 4x_3 &= 20 \\ -x_1 - 3x_2 + 8x_3 &= 4 \\ 5x_1 + 15x_2 - 9x_3 &= 11. \end{aligned}$$

Determine the general solution set to the linear system and describe this set geometrically.⁵

¹The Pauli spin matrices are a set of Hermitian matrices, which are *unitary*. They have found several uses including describing strong interaction symmetries in particle physics and representing logic gates in quantum information theory.

²This statement encapsulates both the symmetric unitary properties of the matrices.

³Here we are using the so-called Levi-Civita symbol. This symbol is used to encode the following commonly encountered information,

$$\epsilon_{ijk} = \begin{cases} 1, & \text{if } (i, j, k) \text{ is } (1, 2, 3), (2, 3, 1) \text{ or } (3, 1, 2), \\ -1, & \text{if } (i, j, k) \text{ is } (3, 2, 1), (1, 3, 2) \text{ or } (2, 1, 3), \\ 0, & \text{if } i = j \text{ or } j = k \text{ or } k = i \end{cases} \quad (3)$$

⁴Here we use the so-called Kronecker delta function, which encodes the, also common, information,

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j \end{cases} \quad (4)$$

⁵Another way to ask this: ‘Are there a set of points in \mathbb{R}^3 where the three previous planes intersect one another? If so, then what geometric object the collection of these points form?’ I hope that it is clear that if there a solution then these points could only form a point, line, or plane, depending the number of free-variables you find by row-reduction.

3. Given the following augmented matrix

$$\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 3 & h & k \end{array} \right].$$

Determine h and k such that the corresponding linear system:⁶

- (a) Is inconsistent.
- (b) Is consistent with infinitely many solutions.
- (c) Is consistent with a unique solution.

4. Suppose a, b, c , and d are constants such the system

$$\begin{aligned} ax_1 + bx_2 &= 0 \\ cx_1 + dx_2 &= 0 \end{aligned}$$

with $a \neq 0$ is consistent for all possible values of f and g . Using row reduction solve for x_1 and x_2 and list any constraints needed, on a, b, c, d , for unique solutions.⁷

5. Given the matrix **A** and the vector **b**.

$$\mathbf{A} = \begin{bmatrix} 5 & 3 \\ -4 & 7 \\ 9 & -2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 22 \\ 20 \\ 15 \end{bmatrix}$$

Are there constants x_1 and x_2 such that **b** can be formed as a linear combination of the columns of **A**? If so then what are they?⁸

⁶Hint: You will not need to find the reduced row-echelon form. Only the row-echelon form is needed.

⁷What we are trying to do here is find conditions on the coefficients a, b, c, d that will guarantee a single solution to the system. Remember that in

⁸we require that to have a unique solution to, $ax = 0$, a must be different from zero.

⁸Another way of asking this: 'Is **b** a linear combination of the columns of **A**?'

Problem 1:

Given,

$$\sigma_1 = \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$\sigma_3 = \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \sigma I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

a) $\sigma_1^{\pm} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$$\sigma_2^{\pm} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\sigma_3^{\pm} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

b)

Note:

$$[BA] = BA - AB = -(AB - BA) = -[A, B]$$

$$[\sigma_i, \sigma_j] = \sigma_i^{\pm} \cdot \sigma_j^{\pm} = \pm 0$$

Thus we only need to consider,

$$\{(1,2), (1,3), (2,3), (1,2,3)\}$$

$$[\sigma_1, \sigma_2] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} - \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Boo.

Again:

$$[\sigma_1, \sigma_2] = \sigma_1 \sigma_2 - \sigma_2 \sigma_1 =$$

$$= [0 \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}, i \begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}] = [0 \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}, i \begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}] =$$

$$= i \begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix} \cdot i \begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix} + 2i \begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix} = 2i \sigma_3$$

and

$$2i \sum_{k=1}^3 e_{2k} \sigma_k = 2i g_{21} \sigma_3 + 2i \sigma_3$$

$$[\sigma_1, \sigma_3] = [0 \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}, i \begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}] =$$

$$= [0 \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}, i \begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}] = [0 \begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}, i \begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}] = 2i \begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix} = 2i \sigma_2$$

and

$$2i \sum_{k=1}^3 e_{3k} \sigma_k = 2i (g_{31} \sigma_1 + g_{32} \sigma_2 + g_{33} \sigma_3) =$$

$$= -2i \sigma_2$$

$$[\sigma_2, \sigma_3] = [0 \begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}, i \begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}] = [0 \begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}, i \begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}] =$$

$$= [0 \begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}, i \begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}] = 2i \begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix} = 2i \sigma_1$$

and

$$2i \sum_{k=1}^3 e_{2k} \sigma_k = 2i g_{21} \sigma_1 + 2i \sigma_1$$

$$2i\sigma_3 : [\sigma_1, \sigma_2] = -[\sigma_2, \sigma_1] = 2i \sum_{k=1}^3 e_{2k} \sigma_k = \\ = -2i \cancel{\sum_{k=1}^3} e_{2k} \sigma_k = 2i\sigma_3 \quad \checkmark$$

$$-2i\sigma_2 : [\sigma_1, \sigma_3] = -[\sigma_3, \sigma_1] = -2i \cancel{\sum_{k=1}^3} e_{3k} \sigma_k = -2i\sigma_2 \quad \checkmark$$

$$2i\sigma_1 : [\sigma_2, \sigma_3] = -[\sigma_3, \sigma_2] = -2i \cancel{\sum_{k=1}^3} e_{3k} \sigma_k = 2i\sigma_1 \quad \checkmark$$

and lastly

$$[\sigma_i, \sigma_i] = 2i \sum_{k=1}^3 e_{kk} \sigma_k = 0 \quad \checkmark$$

Q) Similarly

$$\{ \sigma_i, \sigma_j \} = \sigma_i^2 + \sigma_j^2 = 2\sigma_i^2 = 2I = 2I \delta_{ij} \quad \checkmark$$

and

$$\{ \sigma_i, \sigma_k \} = \{ \sigma_j, \sigma_k \}$$

and

See Previous
Calc.

$$\{ \sigma_i, \sigma_l \} = \sigma_i \sigma_l + \sigma_l \sigma_i = 0$$

$$\{ \sigma_i, \sigma_k \} = \sigma_i \sigma_k + \sigma_k \sigma_i = 0$$

$$\{ \sigma_i, \sigma_3 \} = \sigma_i \sigma_3 + \sigma_3 \sigma_i = 0 \quad \checkmark$$

2. We begin by writing the 3 linear equations (3),(4),(5) as the augmented matrix,

$$\left[\begin{array}{ccc|c} 6 & 18 & -4 & 20 \\ -1 & -3 & 8 & 4 \\ 5 & 15 & -9 & 11 \end{array} \right] \xrightarrow{\begin{array}{l} R_1 \rightarrow R_3 \\ R_3 \rightarrow R_2 \\ R_2 \rightarrow R_1 \end{array}} \left[\begin{array}{ccc|c} -1 & -3 & 8 & 4 \\ 5 & 15 & -9 & 11 \\ 6 & 18 & -4 & 20 \end{array} \right] \sim$$

$$R_2 = 5R_1 + R_2 \quad \left[\begin{array}{ccc|c} -1 & -3 & 8 & 4 \\ 0 & 0 & 40-9 & 11+20 \end{array} \right] \quad R_2 = R_2/44 \quad R_1 = -R_1$$

$$\sim$$

$$R_3 = 6R_1 + R_3 \quad \left[\begin{array}{ccc|c} 0 & 0 & 44 & 44 \end{array} \right] \quad R_3 = R_3/44$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & -8 & -4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array} \right] \quad R_3 = R_3 - R_2 \quad \left[\begin{array}{ccc|c} 1 & 3 & 8 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right] \sim$$

$$R_1 = R_1 + 8R_2 \quad \sim \quad \left[\begin{array}{ccc|c} 1 & 3 & 0 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

which corresponds to the Row equivalent linear system,

$$x_1 + 3x_2 = +4$$

$$x_3 = 1$$

Letting $x_2 = t$ implies that the general solution set is given by,

$$(*) \quad \begin{aligned} x_1 &= -3t + 4 \\ x_2 &= t \quad , \quad t \in \mathbb{R} \\ x_3 &= 1 \end{aligned}$$

With x_1 dependent on the one free variable x_2
(*) parameterizes a 2-D line in 3-D space.

$$4) \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 3 & h & k \end{array} \right] \xrightarrow{R3=R3-3R1} \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & h-9 & k-6 \end{array} \right]$$

corresponds to the linear system,

$$\begin{aligned} x_1 + 3x_2 &= 2 \\ (h-9)x_2 &= k-6 \end{aligned}$$

a. For this system to be consistent with a unique solution,

$$(h-9)x_2 = k-6 \Rightarrow x_2 = \frac{k-6}{h-9}, \text{ assuming } h-9 \neq 0$$

thus $h-9 \neq 0 \Rightarrow h \neq 9$ will yield no free variables and the linear system is consistent with a unique point of intersection of the two lines.

b. For infinitely many solutions (for x_2 to be a free variable) we require that

$$(h-9)x_2 = k-6 \Leftrightarrow 0 \cdot x_2 = 0 \Rightarrow h=9, k=6$$

Thus x_2 is free.

c. For no solutions we require,

$$(k-9)x_2 = k-6 \iff 0 \cdot x_2 = c, c \in \mathbb{R}, c \neq 0.$$

This implies that $k=9$ and $k \neq 6$. Thus the augmented column is a pivot column and the system has no solutions.

1 We have the following augmented matrix representation of (1)+(2),

$$\begin{array}{c} \left[\begin{array}{cc|c} a & b & f \\ c & d & g \end{array} \right] \xrightarrow{\substack{R_2 = aR_2 - bR_1 \\ R_1 = aR_1 - bR_2}} \left[\begin{array}{cc|c} ad - cb & bd - db & df - bg \\ ac - ca & ad - cb & ag - cf \end{array} \right] = \\ = \left[\begin{array}{cc|c} ad - cb & 0 & df - bg \\ 0 & ad - cb & ag - cf \end{array} \right] \xrightarrow{\substack{R_1 = R_1 / (ad - cb) \\ R_2 = R_2 / (ad - cb)}} \left[\begin{array}{cc|c} 1 & 0 & \frac{df - bg}{ad - cb} \\ 0 & 1 & \frac{ag - cf}{ad - cb} \end{array} \right] \end{array} \quad (*)$$

which is equivalent to the linear system,

$$(1') \quad 1 \cdot x_1 + 0 \cdot x_2 = x_1 = \frac{df - bg}{ad - cb}$$

$$(2') \quad 0 \cdot x_1 + 1 \cdot x_2 = \frac{ag - cf}{ad - cb}$$

Note, to do the division at (*) we have assumed
that,

$$ad - bc \neq 0.$$

This is a common statement which places restriction
on a, b, c, d .

5) If $\vec{b} = \begin{bmatrix} 22 \\ 20 \\ 15 \end{bmatrix}$ is a linear combination of the vectors,

$$\vec{a}_1 = \begin{bmatrix} 5 \\ -4 \\ 9 \end{bmatrix} \quad \vec{a}_2 = \begin{bmatrix} 3 \\ 7 \\ -2 \end{bmatrix} \text{ formed from the columns of}$$

$$A = \begin{bmatrix} 5 & 3 \\ -4 & 7 \\ 9 & -2 \end{bmatrix}$$

then there must exist $x_1, x_2 \in \mathbb{R}$ such that

$$\vec{a}_1 \cdot x_1 + \vec{a}_2 \cdot x_2 = \vec{b} \Leftrightarrow \begin{aligned} 5x_1 + 3x_2 &= 22 \\ -4x_1 + 7x_2 &= 20 \\ 9x_1 - 2x_2 &= 15 \end{aligned}$$

To determine if this is true we row reduce the augmented matrix,

$$\left[\begin{array}{cc|c} 5 & 3 & 22 \\ -4 & 7 & 20 \\ 9 & -2 & 15 \end{array} \right] \sim \left[\begin{array}{cc|c} 5 & 3 & 22 \\ 0 & 47 & 188 \\ 0 & -37 & -123 \end{array} \right] \sim$$

$R2 = 5R2 + 4R1$

$R3 = 5R3 - 9R1$

$$\sim \left[\begin{array}{cc|c} 5 & 3 & 22 \\ 0 & 47 & 188 \\ 0 & 0 & 1175 \end{array} \right]$$

$R3 = 47R3 + 37R2$

which corresponds to the linear system,

$$5x_1 + 3x_2 = 22$$

$$47x_2 = 188$$

$$0 \cdot x_2 = 1175 \quad (*)$$

There is no x_2 such that $(*)$ can be satisfied.
Thus, the linear system is inconsistent and \vec{b} is
not a linear combination of \vec{a}_1 and \vec{a}_2 .