# EM fields.

derive fields from potentials:

$$\vec{E} = -\nabla \Phi - \frac{1}{c}\frac{\partial A}{\partial t}$$

$$\vec{B} = \nabla \times \vec{A}$$

x component: $\qquad x_4 = ict$

$$E_1 = -\frac{\partial \Phi}{\partial x_1} - \frac{1}{c}\frac{\partial A_1}{\partial t} = -\frac{1}{i}\frac{\partial A_4}{\partial x_1} + \frac{1}{i}\frac{\partial A_1}{\partial x_4}$$

so

$$iE_1 = \frac{\partial A_1}{\partial x_4} - \frac{\partial A_4}{\partial x_1}$$

$$B_1 = \frac{\partial A_2}{\partial x_3} - \frac{\partial A_3}{\partial x_2}$$

define a tensor

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} \; :\longrightarrow \begin{bmatrix} 0 & B_3 & -B_2 & -iE_1 \\ -B_3 & 0 & B_1 & -iE_2 \\ B_2 & -B_1 & 0 & -iE_3 \\ iE_1 & iE_2 & iE_3 & 0 \end{bmatrix}$$

$\overleftrightarrow{F}$ is antisymmetric

All of Maxwell eqns can be expressed by two eqns:

$$\frac{\partial F_{\lambda\mu}}{\partial x_\nu} + \frac{\partial F_{\mu\nu}}{\partial x_\lambda} + \frac{\partial F_{\nu\lambda}}{\partial x_\mu} = 0$$

and $$\frac{\partial F_{\mu\nu}}{\partial x_\nu} = \frac{4\pi}{c} J_\mu$$

examples $\underline{\lambda=1, \mu=2, \nu=3}$

$$\frac{\partial F_{12}}{\partial x_3} + \frac{\partial F_{23}}{\partial x_1} + \frac{\partial F_{31}}{\partial x_2} = 0$$

$$\frac{\partial B_3}{\partial x_3} + \frac{\partial B_1}{\partial x_1} + \frac{\partial B_2}{\partial x_2} = \nabla \cdot \vec{B} = 0$$

$\underline{\lambda = 1, \mu = 2, \nu = 4}$

$$\frac{\partial F_{12}}{\partial x_4} + \frac{\partial F_{24}}{\partial x_1} + \frac{\partial F_{14}}{\partial x_2} = 0$$

$$\frac{1}{ic}\frac{\partial B_3}{\partial t} + -i\frac{\partial E_2}{\partial x_1} + i\frac{\partial E_1}{\partial x_2} = 0$$

$$\frac{\partial E_1}{\partial x_2} - \frac{\partial E_2}{\partial x_1} = -\frac{1}{c}\frac{\partial B_3}{\partial t} = (\nabla \times \vec{E})_3$$

Field transformations

given $\overleftrightarrow{F}$ in frame $K$ we can boost to another frame $K'$

$$\overleftrightarrow{F}' = \lambda \overleftrightarrow{F} \lambda^{-1} \quad \text{as in matrix notation.}$$

transformed fields go to

$$\vec{E}_\perp' = \gamma\left(\vec{E}_\perp + \frac{1}{c}\vec{v}\times\vec{B}_\perp\right) \qquad E_\parallel' = E_\parallel$$

$$\vec{B}_\perp' = \gamma\left(\vec{B}_\perp - \frac{1}{c}\vec{v}\times\vec{E}_\perp\right) \qquad B_\parallel' = B_\parallel$$

note that some quantities are invariant:

$$\underline{\vec{E}\cdot\vec{B} = \vec{E}'\cdot\vec{B}'}$$

$$\begin{aligned}
\vec{E}'\cdot\vec{B}' &= \vec{E}_\perp'\cdot\vec{B}_\perp' + E_\parallel' B_\parallel' \\
&= \gamma^2\left(\vec{E}_\perp\cdot\vec{B}_\perp - \frac{1}{c^2}(\vec{v}\times\vec{B}_\perp)\cdot(\vec{v}\times\vec{E}_\perp) + 0 + 0\right) + E_\parallel B_\parallel \\
&= \gamma^2\left(\vec{E}_\perp\cdot\vec{B}_\perp - \frac{1}{c^2}\left(v^2\vec{E}_\perp\cdot\vec{B}_\perp - \cancel{(\vec{v}\cdot\vec{E}_\perp)(\vec{v}\cdot\vec{B}_\perp)}\right)\right) + E_\parallel B_\parallel \\
&= \gamma^2\left(1-\frac{v^2}{c^2}\right)\vec{E}_\perp\cdot\vec{B}_\perp + E_\parallel B_\parallel = \vec{E}\cdot\vec{B}
\end{aligned}$$

$\underline{E^2 - B^2 = (E')^2 - (B')^2}$ also

for light, invariants are zero. $E\cdot B = 0$ $\quad E^2 = B^2$

if $B = 0$ in one frame, $\rightarrow E', B'$ in another.

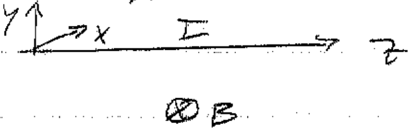
can't find frame where $E' = 0$

vice-versa is true.

Wire problem with field transformation:

in lab frame, local $\vec{B}$ field is in $+\hat{x}$ direction $\quad B_0\hat{x}$

$\vec{E} = 0$

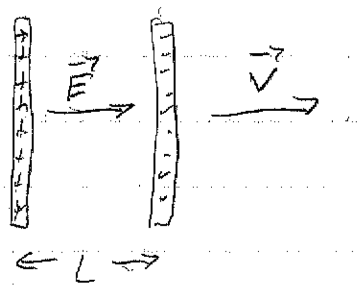



boost by vel. $u$ in $z$-direction:

$$E'_\perp = \gamma_u \cdot \frac{1}{c}\,\vec{u} \times \vec{B}_\perp$$

$$= +\hat{y}\,\frac{u}{c}\,B_0\,\gamma_u \quad \text{same as before.}$$

examples of field transf.




boost

$Q/A$ unchanged

$L \to L'$

but $E$ is indep of $L$

$\therefore\ E'_\parallel = E_\parallel$

$B'_\perp = 0$ since $\vec{v} \parallel \vec{E}_\parallel$

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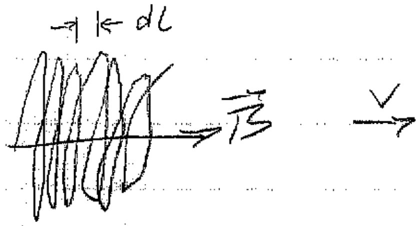
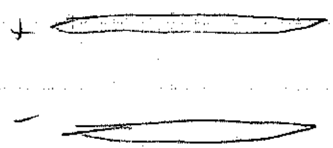



$$\left.\begin{aligned} dL' &= dL/\gamma \\ \text{but } j' &= \gamma j \end{aligned}\right\} \text{cancel}$$

$\therefore\ B'_\parallel = B_\parallel$




$Q/A' = \gamma\, Q/A$

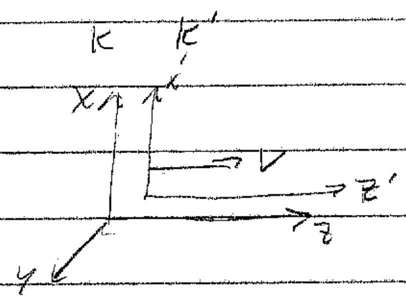
$\therefore\ \vec{E}'_\perp = \gamma\,\vec{E}_\perp$

$$\vec{B}'_\perp = -\gamma\,\frac{1}{c}\,\vec{v} \times \vec{E}_\perp$$

$$= -\frac{1}{c}\,\vec{v} \times \vec{E}'_\perp$$

Transformation of EM wave:

$\vec{E}(\vec{r},t) = \hat{x} E_0 e^{i(kz-\omega t)}$




$\vec{B}(\vec{r},t) = \hat{y} E_0 e^{i(kz-\omega t)}$

view from moving frame

first check $z, t$:

$z' = \gamma z' + \gamma v t'$

$ict = +i\beta\gamma z' + i\gamma c t'$

$t = \gamma(t' + v z'/c^2)$

sign check: fixed $z'$ moves forward in K

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \gamma & i\beta\gamma \\ & & -i\beta\gamma & \gamma \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ z' \\ ict' \end{bmatrix}$$

$$kz - \omega t = k\gamma(z' + vt') - \omega\gamma(t' + vz'/c^2)$$
$$= \gamma(k - v\omega/c^2) z' - \gamma(\omega - vk) t'$$

Now it appears we have a new $k' = \gamma\left(k - \frac{v\omega}{c^2}\right)$

and $\omega' = \gamma(\omega - vk)$

so

$kz - \omega t = k'z' - \omega' t'$

this is equivalent to saying phase is invariant:

$\phi = \phi'$

$\phi$ is a scalar

$r_\mu$ is a 4 vector. $(\vec{r}, ict)$

$\therefore$ exists another 4 vector $r_\mu k_\mu = \phi$

$k_\mu \to (\vec{k}, ik_0)$ $\quad k_0 = \omega/c$

$r_\mu k_\mu = \vec{r}\cdot\vec{k} - k_0 c t = \vec{k}\cdot\vec{r} - \omega t$

we can get $k_\mu'$ directly: $k_\mu' = \lambda_{\mu\nu} k_\nu$

check $k_z' = \gamma k_z - k_0\beta\gamma = \gamma(k_z - v\omega/c^2)$

since $\omega = ck$ (and $\omega' = ck'$ still)

$$\omega' = \gamma(\omega - v\omega/c) = \gamma(1 - v/c)\,\omega = \sqrt{\frac{1 - v/c}{1 + v/c}}\,\omega$$

this is the Doppler shift (relativistic) if $v > 0$ $\omega' < \omega$

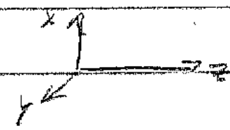
Fields?

fields are $\perp$ to boost.

$$\vec{E}_\perp' = \vec{E}_x' = \gamma(\vec{E}_x + \tfrac{1}{c}\vec{v} \times \vec{B}_y)$$

$$= \gamma \vec{E}_x (1 - v/c)$$ same direction,

amplitude changes as $\omega'/\omega$




$$\vec{B}_\perp' = \vec{B}_y' = \gamma(B_y - \tfrac{1}{c}\vec{v} \times \vec{E}_x)$$

$$= \gamma B_y (1 - v/c)$$

As $v \to c$ $\omega \to 0$ and ampl. $\to 0$