Homework #5 Solutions

1. Calculate the following integrals

a) $\int x^3 cos(5x)dx$

$$\begin{array}{c|c} u & dv \\ \hline x^3 & cos(5x) \\ 3x^2 & \frac{sin(5x)}{5} \\ 6x & \frac{-cos(5x)}{25} \\ 6 & \frac{-sin(5x)}{125} \\ 0 & \frac{cos(5x)}{625} \end{array} =$$

$$= \frac{x^3 \cdot sin(5x)}{5} + \frac{3x^2 \cdot cos(5x)}{25} - \frac{6x \cdot sin(5x)}{125} - \frac{6 \cdot cos(5x)}{625} + c$$

b)$\int x^2 sin(2x^3)dx$

$$\substack{u=2x^3 \\ du=6x^2} \quad \Rightarrow \quad \frac{1}{6}\int sin(u)du =$$

$$= \frac{1}{6}(-cos(u)) + c = \frac{-cos(2x^3)}{6} + c$$

c)$\int e^{ax} cos(bx)dx$, $\begin{array}{c|c} u & dv \\ \hline e^{ax} & cos(bx) \\ ae^{ax} & \frac{1}{b}sin(bx) \\ a^2e^{ax} & \frac{1}{b^2}cos(bx) \end{array} \Rightarrow$

$$\begin{aligned} \int e^{ax}cos(bx)dx &= \frac{e^{ax}}{b}sin(bx) + \frac{ae^{ax}}{b^2}cos(bx) - \frac{a^2}{b^2}\int e^{ax}cos(bx)dx \\ \int e^{ax}cos(bx)dx + \frac{a^2}{b^2}\int e^{ax}cos(bx)dx &= \frac{e^{ax}}{b}sin(bx) + \frac{ae^{ax}}{b^2}cos(bx) - \frac{a^2}{b^2}\int e^{ax}cos(bx)dx \\ (1+\frac{a^2}{b^2})\int e^{ax}cos(bx)dx &= \frac{e^{ax}}{b}sin(bx) + \frac{ae^{ax}}{b^2}cos(bx) \\ \int e^{ax}cos(bx)dx &= \frac{1}{1+\frac{a^2}{b^2}}\left[\frac{e^{ax}}{b}sin(bx) + \frac{ae^{ax}}{b^2}cos(bx)\right] = \\ &= \frac{1}{a^2+b^2}\left[be^{ax}sin(bx) + ae^{ax}cos(bx)\right] + c \end{aligned}$$

d)$\int_0^{2\pi} sin(nx)cos(mx)dx$ where $m, n \in Z$

trig identity: $sinx \cdot cosy = \frac{1}{2}sin(x+y) + \frac{1}{2}sin(x-y)$

$\frac{1}{2}\int_0^{2\pi}\left[sin(m+n)x + sin(m-n)x\right]dx$

$$\text{if } m \neq n \quad \Rightarrow \quad \frac{1}{2}\left[\frac{cos(m+n)x}{m+n} + \frac{cos(m-n)}{m-n}\right]_0^{2\pi}$$

$$= \frac{1}{2}\left[\frac{cos(m+n)2\pi}{m+n} + \frac{cos(m-n)2\pi}{m-n} - \frac{cox(0)}{m+n} - \frac{cos(0)}{m-n}\right]$$

$$
\begin{aligned}
&= \frac{1}{2}\left[\frac{1}{m+n}+\frac{1}{m-n}-\frac{1}{m+n}-\frac{1}{m-n}\right]\\
&= 0\\
\text{if } m=n \quad &\Rightarrow \frac{1}{2}\int_0^{2\pi}\left[sin(m+n)x+sin(m-n)x\right]dx\\
&= \frac{1}{2}\int_0^{2\pi}\left[sin(2nx)+0\right]dx\\
&= \frac{1}{2}\left[\frac{-cos(2nx)}{2n}\right]_0^{2\pi}\\
&= \frac{1}{2}\left[-cos(2\pi 2n)+cos(0)\right]\\
&= 0
\end{aligned}
$$

2. given: $\hat{\imath} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}, \qquad \hat{\jmath} = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$

a) Show that any vector for R^2 can be created from a linear combination of $\hat{\imath}$ and $\hat{\jmath}$. That is, given: $\hat{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1\hat{\imath}+c_2\hat{\jmath}$ show that c_1 and c_2 can be found in terms of x_1 and x_2.

$$
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} + c_2\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} \Leftrightarrow \begin{array}{ll} x_1 = c_1\left(\frac{\sqrt{2}}{2}\right)+c_2\left(\frac{\sqrt{2}}{2}\right) & (1) \\ x_2 = c_1\left(\frac{\sqrt{2}}{2}\right)+c_2\left(\frac{\sqrt{2}}{2}\right) & (2) \end{array}
$$

Adding (1) and (2), we get

$$
x_1+x_2 = 2c_1\left(\frac{\sqrt{2}}{2}\right) \Leftrightarrow c_1 = \frac{x_1+x_2}{\sqrt{2}}
$$

Multipllying (1) by -1 and adding (1) and (2)

$$
x_2-x_1 = 2c_2\left(\frac{\sqrt{2}}{2}\right) \Leftrightarrow c_2 = \frac{x_2-x_1}{\sqrt{2}}
$$

3. Given: $f(x) = cos(x)$ and $g(x) = sin(x)$
and $f(x)\cdot g(x) = \int_{-\pi}^{\pi} f(x)g(x)dx$ show that
a) $f(nx)\cdot f(mx) = \pi\delta_{nm}$ where $n, m \in Z$

$$
\begin{aligned}
f(nx)\cdot f(mx) &= \int_{-\pi}^{\pi} cos(nx)cos(mx)dx\\
\text{Assuming } n\neq m \quad &\Rightarrow \int_{-\pi}^{\pi}\frac{1}{2}[cos(n-m)x+cos(n+m)x]dx\\
&= \frac{1}{2}\left[\frac{sin(n-m)x}{(n-m)}+\frac{sin(n+m)x}{(n+m)}\right]_{-\pi}^{-\pi}
\end{aligned}
$$

$$
\begin{aligned}
& \text{because } n,m \in Z, n \pm m \in Z \text{ and } sin(k\pi) = 0 \text{ if } k \in Z \\
&= 0 \\
\text{Assuming} n = m \quad &\Rightarrow \int_{-\pi}^{\pi} cos(nx)cos(mx)dx = \int_{-\pi}^{\pi} cos^2(nx)dx \\
&= \int_{-\pi}^{\pi} \left[\frac{1}{2} + \frac{1}{2}cos(2nx)\right] dx \\
&= \left[\frac{1}{2}x + \frac{1}{4n}sin(2nx)\right]_{-\pi}^{\pi} \\
&= \frac{1}{2}[\pi - (-\pi)] + 0 \\
&= \pi
\end{aligned}
$$

Thus,

$$
f(nx) \cdot f(mx) = \begin{cases} 0 & m \neq n \\ \pi & m = n \end{cases} = \pi\delta_{mn}
$$

b) $g(nx) \cdot g(mx) = \pi\delta_{mn}$ where $n, m \in Z$

$$
\begin{aligned}
g(nx) \cdot g(mx) &= \int_{-\pi}^{\pi} sin(nx)sin(mx)dx \\
\text{Assume } n \neq m \quad &\Rightarrow \int_{-\pi}^{\pi} \frac{1}{2}[cos(n-m)x - cox(n+m)x]dx \\
&= \frac{1}{2}\left[\frac{sin(n-m)x}{n-m} - \frac{sin(n+m)x}{n+m}\right]_{-\pi}^{\pi} \\
&= \frac{1}{2}[0] = 0 \\
\text{Assume } n = m \quad &\Rightarrow \int_{-\pi}^{\pi} sin^2(nx)dx \\
&= \int_{-\pi}^{\pi} \left[\frac{1}{2} - \frac{1}{2}cos(2nx)\right] dx \\
&= \left[\frac{1}{2} - \frac{1}{4n}sin(2nx)\right]_{-\pi}^{\pi} \\
&= \frac{1}{2}[\pi - (-\pi)] = \pi \\
g(nx) \cdot g(mx) &= \begin{cases} 0 & m \neq n \\ \pi & n = m \end{cases} = \pi\delta_{nm}
\end{aligned}
$$

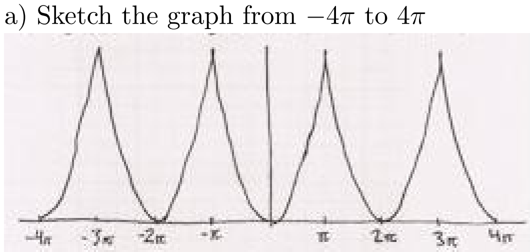
c) $f(nx) \cdot g(mx) = 0, \quad n, m \in Z$

$$
\begin{aligned}
f(nx) \cdot g(mx) &= \int_{-\pi}^{\pi} cos(nx)sin(mx)dx \\
\text{Assume } n \neq m \quad &\Rightarrow \int_{-\pi}^{\pi} \frac{1}{2}[sin(n-m)x + sin(n+m)x]dx
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\left[\frac{-cos(n-m)x}{n-m} - \frac{cos(n+m)}{n+m}\right]_{-\pi}^{\pi} \\
&= \frac{1}{2}\left[\frac{(-1)^n}{n-m} - \frac{(-1)^n}{n-m} + \frac{(-1)^n}{m+n} - \frac{(-1)^n}{n+m}\right] \\
&= 0 \\
\text{Assume } n = m \quad &\Rightarrow \int_{-\pi}^{\pi} cos(nx)sin(nx)dx \\
&= \int_{-\pi}^{\pi} \frac{1}{2}[sin(n-n)x + sin(n+n)x]dx \\
&= \int_{-\pi}^{\pi} \frac{1}{2} sin(2nx)dx \\
&= \frac{-1}{4n}\left[cos(2nx)\right]_{-\pi}^{\pi} \\
&= 0 \\
f(nx)\cdot g(mx) &= 0 \quad \text{no matter the choice of m and n}
\end{aligned}
$$

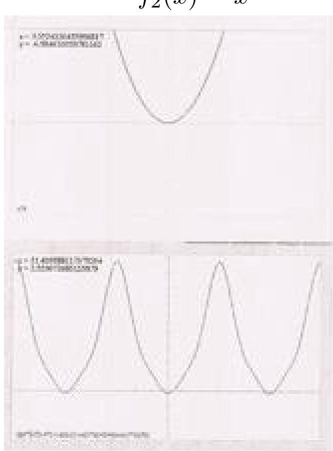
4. Let $f(x) = x^2, \quad (-\pi < x < \pi)$ be a 2π-periodic function

a) Sketch the graph from -4π to 4π



b) Graph $f_1(x) = \frac{\pi^2}{3} - 4\left(cos(x) - \frac{cos(2x)}{4} + \frac{cos(3x)}{9}\right)$

$f_2(x) = x^2$



5. a) A fourier series is a method of decomposing periodic functions in terms of sines and cosines with discrete frequencies. Typically, a fourier series is used to understand a function's frequency spectrum and because of this, appears heavily in signal analysis. However, as a tool, it is very powerful and appears in many applications having to do with P.D.E's.
b) Near the points of discontinuity there is a ringing, or Gibb's phenomenon. At the point of discontinuity, the fourier series will average the left- and right-hand limits of the function.
c) The sum of periodic functions is always periodic. However, the infinite sum of continuous functions may be discontinuous.