Homework #3 Solutions

1. Using $\vec{v}_1, \vec{v}_2, \vec{v}_3$ construct the matrix,

$$V = [\vec{v}_1, \vec{v}_2, \vec{v}_3] = \begin{bmatrix} 1 & -5 & 1 \\ -1 & 7 & 1 \\ -3 & 8 & h \end{bmatrix}$$

The columns of V are linearly independent if and only if $V\vec{x} = \vec{0}$ has only the trivial solution. Thus,

$$\left[\begin{array}{ccc|c} 1 & -5 & 1 & 0 \\ -1 & 7 & 1 & 0 \\ -3 & 8 & h & 0 \end{array}\right] \sim_{R2=R1+R2}^{R3=3R3+R1} \left[\begin{array}{ccc|c} 1 & -5 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -7 & 3+h & 0 \end{array}\right] \sim^{R3=2R3+7R2}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -5 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2h+20 & 0 \end{array}\right]$$

shows that $V\vec{x} = \vec{0}$ has nontrivial solutions for $2h + 20 = 0 \Leftrightarrow h = -10$. Hence, if $h = -10$ then $V\vec{x} = \vec{0}$ has nontrivial solutions and if $V\vec{x} = \vec{0}$ has nontrivial solutions then the columns of V are linearly dependent.

2. Let $A = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix}, \vec{w} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

a. If $\vec{w} \in ColA$ then $\vec{w} \in span\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$. Check by row reduction

$$\begin{bmatrix} A & \vec{w} \end{bmatrix} = \left[\begin{array}{ccc|c} -8 & -2 & -9 & 2 \\ 6 & 4 & 8 & 1 \\ 4 & 0 & 4 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} -8 & -2 & -9 & 2 \\ 0 & 20 & 10 & 20 \\ 0 & -2 & -1 & -2 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc|c} -8 & -2 & -9 & 2 \\ 0 & 20 & 10 & 20 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$ x_3 is free and x_1, x_2 are uniquely determined in terms of x_3.

Thus, there exists x_1, x_2, x_3 such that $x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 = \vec{w}$ and $\vec{w} \in ColA$.

b.

$$A\vec{w} = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

thus $\vec{w} \in NulA$

3.

$$A = \begin{bmatrix} 2 & -3 & 6 & 2 & -5 \\ -2 & 3 & -3 & -3 & -4 \\ 4 & -6 & 9 & 5 & 9 \\ -2 & 3 & 3 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & -3 & 1 & -1 \\ 0 & 0 & 9 & -2 & 6 \end{bmatrix} \sim$$

$$\sim \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & -3 & 1 & -1 \\ 0 & 0 & 9 & -2 & 6 \end{bmatrix} \sim \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 13 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix} \sim$$

$$\sim \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = B$$

a. $B \Rightarrow A\vec{x} = 0$ has the general solution set.

$$\begin{aligned} x_4 &= -3x_5 \\ x_3 &= (x_4 - x_5)/3 = (-3x_5 - x_5)/3 = \frac{-4}{3}x_5 \\ x_1 &= \frac{1}{2}(3x_2 - 6x_3 - 2x_4 - 5x_5) = \frac{1}{2}(3x_2 - 6(-4x_5) - 2(-3x_5) - 5x_5) = \\ &= \frac{1}{2}(3x_2 + 8x_5 + 6x_5 - 5x_5) = \frac{3}{2}x_2 + \frac{9}{2}x_5 \\ x_2 &= \text{free} \\ x_5 &= \text{free} \\ &\Rightarrow \quad \vec{x} = x_2 \begin{bmatrix} 3/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -9/2 \\ 0 \\ -4/3 - 3 \\ 1 \end{bmatrix} \qquad x_2, x_5 \in \Re \end{aligned}$$

Thus the basis for NulA is

$$B_{null} = \left\{ \begin{bmatrix} 3/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -9/2 \\ 0 \\ -4/3 \\ -3 \\ 1 \end{bmatrix} \right\}$$

and $dim(NulA) = dimB_{null} = 2$

b. $B \Rightarrow$ that the basis for the column space of A is the pivot columns $\vec{a}_1, \vec{a}_3, \vec{a}_4$ of A

$$B_{ColA} = \left\{ \begin{bmatrix} 2 \\ -2 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ 7 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 5 \\ -4 \end{bmatrix} \right\}$$

and $dim(ColA) = dimB_{ColA} = 3$

c. $B \Rightarrow$ the basis for RowA is given as

$$B_{RowA} = \left\{ \begin{matrix} [2, -3, 6, 2, 5] \\ [0, 0, 3, -1, 1] \\ [0, 0, 0, 1, 3] \end{matrix} \right\}$$

$dim(RowA) = dimB_{RowA} = 3$

4.
a) There are 3 vectors in $V = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$
b) There are infinitely many vectors in spanV. Each is of the form
$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$ where $c_1, c_2, c_3 \in \Re$
c) If $\vec{w} \in spanV$ then$\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$ which is the same as asking if

$$V\vec{c} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ -1 & 3 & 6 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \vec{w}$$

has a solution.

$$\left[\begin{array}{c|c} V & \vec{w} \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ -1 & 3 & 6 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 5 & 10 & 5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since there are no inconsistent rows there exists a solution. Since there is a solution we can conclude that $\vec{w} \in spanV$

5.
a.

$$\begin{aligned} y_1(t) &= cos(wt) \Rightarrow y_1''(t) = -w^2cos(wt) \\ y_2(t) &= sin(wt) \Rightarrow y_2''(t) = -w^2sin(wt) \\ &\Rightarrow my_1'' + ky_1 = -mw^2cos(wt) + k \cdot cos(wt) \\ &= cos(wt)(-mw^2 + k) = cos(wt)(-k + k) = 0 \end{aligned}$$

and

$$my_2'' + ky_2 = sin(wt)(-mw^2 + k) = 0$$

b. Let $y \in span\{y_1, y_2\}$ then $y(t) = c_1cos(wt) + c_3sin(wt)$ and

$$my'' + ky = c_1cos(wt)(-w^2m + k) + c_2sin(wt)(-w^2m + k)$$

thus a function in $span\{y_1, y_2\}$ are solutions