which states that the total number of cars is constant for all time. The constant could be evaluated if *either* the initial number of cars N_0 or the initial density $\rho(x, 0)$ were known:

$$\int_{-\infty}^{\infty} \rho(x, t)\, dx = N_0 = \int_{-\infty}^{\infty} \rho(x, 0)\, dx.$$

The integral conservation law, equation 60.3, will be expressed as a **local conservation law**, valid at each position of the roadway. We will do so in three *equivalent* ways.* In all three, the endpoints of the segment of the roadway, $x = a$ and $x = b$, are considered as additional independent variables. Thus, the full derivative with respect to time in equation 60.3 must be replaced by a partial derivative,

$$\boxed{\frac{\partial}{\partial t}\int_a^b \rho(x, t)\, dx = q(a, t) - q(b, t),} \qquad \textbf{(60.4a)}$$

since the derivation of equation 60.3 assumed the positions $x = a$ and $x = b$ are fixed in time. In other words, $\partial/\partial t$ means d/dt holding a and b fixed in time. In the first derivation, we investigate a small segment of the roadway. Rough approximations are made, which yield the correct result. However, those who are not convinced by our first approximation should be patient, as we will shortly improve the derivation.

(1) Consider the integral conservation of cars over a small interval of highway from $x = a$ to $x = a + \Delta a$. Thus from equation 60.4a,

$$\frac{\partial}{\partial t}\int_a^{a+\Delta a} \rho(x, t)\, dx = q(a, t) - q(a + \Delta a, t).$$

Divide by $-\Delta a$ and take the limit as $\Delta a \to 0$:

$$\lim_{\Delta a \to 0} \frac{\partial}{\partial t} \frac{1}{-\Delta a}\int_a^{a+\Delta a} \rho(x, t)\, dx = \lim_{\Delta a \to 0} \frac{q(a, t) - q(a + \Delta a, t)}{-\Delta a}. \qquad \textbf{(60.4b)}$$

The right-hand side of equation 60.4b is exactly the definition of the derivative of $q(a, t)$ with respect to a (properly a partial derivative should be used, since t is fixed), $(\partial/\partial a)q(a, t)$. On the left-hand side of equation 60.4b the limit can be performed in two equivalent ways:

(a) The integral is the area under the curve $\rho(x, t)$ between $x = a$ and $x = a + \Delta a$. Since Δa is small, the integral can be approximated by one rectangle; as shown in Fig. 60-2. The number of cars between a

*In these derivations we will assume $q(x, t)$, $\rho(x, t)$ and $u(x, t)$ are continuous functions of x and t. In later sections (see Sec. 77) we will find it necessary to relax these assumptions. We will find that, although equation 60.2 is always valid, later results of this section are valid only in regions in which the traffic variables are continuous functions of x and t.

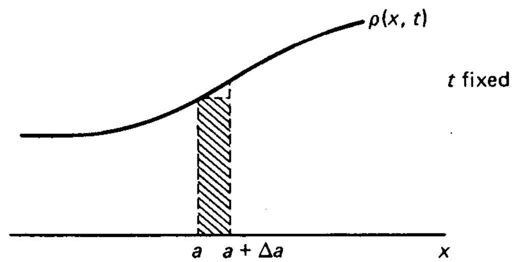

Figure 60-2.

and $a + \Delta a$ can be approximated by the length of roadway Δa times the traffic density at $x = a$, $\rho(a, t)$. Thus,

$$-\frac{1}{\Delta a}\int_a^{a+\Delta a} \rho(x, t)\, dx \approx -\rho(a, t).$$

In a subsequent derivation, (2), we show that the error vanishes as $\Delta a \to 0$. Consequently we derive from equation 60.4b that

$$\boxed{\frac{\partial}{\partial t}\rho(a, t) + \frac{\partial}{\partial a}q(a, t) = 0.} \qquad \textbf{(60.5a)}$$

(b) On the other hand, introduce the function $N(\bar{x}, t)$, the number of cars on the roadway between any fixed position x_0 and the variable position $\bar{x}$,

$$N(\bar{x}, t) \equiv \int_{x_0}^{\bar{x}} \rho(x, t)\, dx.$$

Then, the average number of cars per mile between a and $a + \Delta a$ is

$$-\frac{1}{\Delta a}\int_a^{a+\Delta a} \rho(x, t)\, dx = \frac{N(a + \Delta a, t) - N(a, t)}{-\Delta a}.$$

In the limit as $\Delta a \to 0$, the right-hand side is $-\partial N(a, t)/\partial a$. Using the definition of $N(a, t)$, again from the Fundamental Theorem of Calculus,

$$\frac{\partial N(a, t)}{\partial a} = \rho(a, t).$$

Thus the left-hand side of equation 60.4b again equals $-(\partial/\partial t)\rho(a, t)$. By either method, (a) or (b), equation 60.5a follows. Since equation 60.5a holds for all values of a, it is more appropriate to replace a by x, in which case

$$\boxed{\frac{\partial \rho(x, t)}{\partial t} + \frac{\partial}{\partial x}[q(x, t)] = 0,} \qquad \textbf{(60.5b)}$$