Advanced Engineering Mathematics		**Homework Six**
Differential Equations : Sturm-Liouville Problems, Power Series, Conservations Laws, PDE		
Text: 5.1, 5.7, 12.1	Lecture Notes : ODE Review and 13	Lecture Slides: N/A

Quote of Homework Seven
In life there an infinitely-many directions and each one is permitted.
CSM Professor Emeritus: John DeSanto - Mathematical Physics (2007)

1. Sturm-Liouville Problems

Recall the Sturm-Liouville eigenproblem given by,

(1) $$Lu = \frac{1}{w(x)}\left(-\frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + q(x)u\right) = \lambda u, \ \lambda \in \mathbb{C}$$

whose nontrivial eigenfunctions must satisfy the boundary conditions,

(2) $$k_1 u(a) + k_2 u'(a) = 0$$

(3) $$l_1 u(b) + l_2 u'(b) = 0.$$

1.1. **Orthogonality of Solutions.** Let (λ_1, u_1) and (λ_2, u_2) be two different eigenvalue/eigenfunction pairs. Show that u_1 and u_2 are orthogonal. That is, show that $\langle u_1, u_2\rangle = 0$ with respect to the inner-product defined by $\langle f, g\rangle = \int_a^b f(x)g(x)w(x)dx$.

1.2. **Bessel's Equation.** Show that if $p(x) = x$, $q(x) = \nu^2/x$ and $w(x) = x/\lambda$ then (1) becomes $x^2u'' + xu' + (x^2 - \nu^2)u = 0$, which is known as Bessel's equation of order ν.

1.3. **Fourier Bessel Series.** A solution to Bessel's equation is for $\nu = n \in \mathbb{N}$,

(4) $$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!(n+m)!}, \ n = 1, 2, 3, \ldots$$

which is called Bessel's function of the first-kind of order n. Since these functions manifest from a SL problem they naturally orthogonal and have an orthogonality condition,

(5) $$\langle J_n(xk_{n,m}), J_n(xk_{n,i})\rangle = \int_0^R xJ_n(xk_{n,m})J_n(xk_{n,i})dx = \frac{\delta_{mi}}{2}\left[RJ_{n+1}(k_{n,i}R)\right]^2.$$

Using this show that the coefficients in the Fourier-Bessel series,

(6) $$f(x) = \sum_{m=1}^{\infty} a_m J_n(k_{n,m}x),$$

are given by,

(7) $$a_i = \frac{2}{R^2 J_{n+1}^2(k_{n,m}R)} \int_0^R xJ_n(k_{n,i}R)f(x)dx, \ \ i = 1, 2, 3, \ldots$$

2. Power-Series Solutions to ODE's and Hyperbolic Trigonometric Functions

Consider the ordinary differential equation:

(8) $$y'' - y = 0$$

2.1. **General Solution - Standard Form.** Show that the solution to (8) is given by $y(x) = c_1e^x + c_2e^{-x}$.

2.2. **General Solution - Nonstandard Form.** Show that $y(x) = b_1 \sinh(x) + b_2 \cosh(x)$ is a solution to (8) where $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$ and $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$.

2.3. **Conversion from Standard to Nonstandard Form.** Show that if $c_1 = \dfrac{b_1 + b_2}{2}$ and $c_2 = \dfrac{b_1 - b_2}{2}$ then $y(x) = c_1e^x + c_2e^{-x} = b_1 \cosh(x) + b_2 \sinh(x)$.

2.4. **Relation to Power-Series.** Assume that $y(x) = \sum_{n=0}^{\infty} a_n x^n$ to find the general solution of (8) in terms of the hyperbolic sine and cosine functions. [1]

3. Conservation Laws in One-Dimension

Recall that the conservation law encountered during the derivation of the heat equation was given by,

$$\frac{\partial u}{\partial t} = -\kappa \nabla \phi, \tag{10}$$

which reduces to

$$\frac{\partial u}{\partial t} = -\kappa \frac{\partial \phi}{\partial x}, \quad \kappa \in \mathbb{R} \tag{11}$$

in one-dimension of space.[2] In general, if the function $u = u(x,t)$ represents the density of a physical quantity then the function $\phi = \phi(x,t)$ represents its flux. If we assume the ϕ is proportional to the negative gradient of u then, from (11), we get the one-dimensional heat/diffusion equation (**??**).[3]

3.1. **Transport Equation.** Assume that ϕ is proportional to u to derive, from (11), the convection/transport equation, $u_t + cu_x = 0 \;\; c \in \mathbb{R}$.

3.2. **General Solution to the Transport Equation.** Show that $u(x,t) = f(x - ct)$ is a solution to this PDE.

3.3. **Diffusion-Transport Equation.** If both diffusion and convection are present in the physical system then the flux is given by, $\phi(x,t) = cu - du_x$, where $c, d \in \mathbb{R}^+$. Derive from, (11), the convection-diffusion equation $u_t + cu_x - du_{xx} = 0$.

3.4. **Convection-Diffusion-Decay.** If there is also energy/particle loss proportional to the amount present then we introduce to the convection-diffusion equation the term λu to get the convection-diffusion-decay equation,[4]

3.5. **General Importance of Heat/Diffusion Problems.** Given that,

$$u_t = Du_{xx} - cu_x - \lambda u. \tag{12}$$

Show that by assuming, $u(x,t) = w(x,t)e^{\alpha x - \beta t}$, (12) can be transformed into a heat equation on the new variable w where $\alpha = c/(2D)$ and $\beta = \lambda + c^2/(4D)$.[5]

4. Some Solutions to common PDE

Show that the following functions are solutions to their corresponding PDE's.

4.1. **Right and Left Travelling Wave Solutions.** $u(x,t) = f(x - ct) + g(x + ct)$ for the 1-D wave equation.

4.2. **Decaying Fourier Mode.** $u(x,t) = e^{-4\omega^2 t}\sin(\omega x)$ where $c = 2$ for the 1-D heat equation.

4.3. **Radius Reciprocation.** $u(x,y,z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$ for the 3-D Laplace equation.

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[1] The hyperbolic sine and cosine have the following Taylor's series representations centred about $x = 0$,

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \qquad \sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}. \tag{9}$$

It is worth noting that these are basically the same Taylor series as cosine/sine with the exception that the signs of the terms do not alternate. From this we can gather a final connection for wrapping all of these functions together. If you have the Taylor series for the exponential function and extract the even terms from it then you have the hyperbolic cosine function. Taking the hyperbolic cosine function and alternating the sign of its terms gives you the cosine function. Extracting the odd terms from the exponential function gives the same statements for the hyperbolic sine and sine functions. The reason these functions are connected via the imaginary number system is because when i is raised to integer powers it will produce these exact sign alternations. So, if you remember $e^x = \sum_{n=0}^{\infty} x^n/n!$ and $i = \sqrt{-1}$ then the rest (hyperbolic and non-hyperbolic trigonometric functions) follows!

[2] When discussing heat transfer this is known as Fourier's Law of Cooling. In problems of steady-state linear diffusion this would be called Fick's First Law. In discussing electricity u could be charge density and q would be its flux.

[3] AKA Fick's Second Law associated with linear non-steady-state diffusion.

[4] The u_{xx} term models diffusion of energy/particles while u_x models convection, u models energy/particle loss/decay. The final term should not be surprising! Wasn't the appropriate model for radioactive/exponential decay $Y' = -\alpha^2 Y$?

[5] This shows that the general PDE (12) can be solved using heat equation techniques.

4.4. **Driving/Forcing Affects.** $u(x,y) = x^4 + y^4$ where $f(x,y) = 12(x^2+y^2)$ for the 2-D Poisson equation.

Note: The PDE in question are,

- Laplace's equation : $\triangle u = 0$
- Poisson's equation : $\triangle u = f(x,y,z)$
- Heat/Diffusion Equation : $u_t = c^2 \triangle u$
- Wave Equation : $u_{tt} = c^2 \triangle u$

and can be found on page 563 of Kryszig. The following will outline some common notations. It is assumed all differential operators are being expressed in Cartesian coordinates.[6]

- Notations for partial derivatives,

$$\frac{\partial u}{\partial x} = u_x = \partial_x u \tag{13}$$

- Nabla the differential operator,

$$\nabla = \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} \tag{14}$$

- Gradient of a scalar function,

$$\nabla u = \begin{bmatrix} \partial_x u \\ \partial_y u \\ \partial_z u \end{bmatrix} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} \tag{15}$$

- Divergence of a vector,

$$\nabla \cdot \boldsymbol{v} = \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \partial_x v_1 + \partial_y v_2 + \partial_z v_3 \tag{16}$$

- Curl of a vector,

$$\nabla \times \mathbf{v} = \begin{bmatrix} \partial_y v_3 - \partial_z v_2 \\ \partial_z v_1 - \partial_x v_3 \\ \partial_x v_2 - \partial_y v_1 \end{bmatrix} \tag{17}$$

- Notations for the Laplacian,

$$\triangle u = \nabla \cdot \nabla u = \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} \cdot \begin{bmatrix} \partial_x u \\ \partial_y u \\ \partial_z u \end{bmatrix} \tag{18}$$

$$= \partial_{xx} u + \partial_{yy} u + \partial_{zz} u \tag{19}$$

$$= u_{xx} + u_{yy} + u_{zz} \tag{20}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \tag{21}$$

[6]Of course others have worked out the common coordinate systems, which requires some elbow grease and the multivariate chain rule. Those interested in the results can find them at Nabla in Cylindrical and Spherical